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Dislocations in second strain gradient elasticity

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Abstract

A second strain gradient elasticity theory is proposed based on first and second gradients of the strain tensor. Such a theory is an extension of first strain gradient elasticity with double stresses. In particular, the strain energy depends on the strain tensor and on the first and second gradient terms of it. Using a simplified but straightforward version of this gradient theory, we can connect it with a static version of Eringen's nonlocal elasticity. For the first time, it is used to study a screw dislocation and an edge dislocation in second strain gradient elasticity. By means of this second gradient theory it is possible to eliminate both strain and stress singularities. Another important result is that we obtain nonsingular expressions for the force stresses, double stresses and triple stresses produced by a straight screw dislocation and a straight edge dislocation. The components of the force stresses and of the triple stresses have maximum values near the dislocation line and are zero there. On the other hand, the double stresses have maximum values at the dislocation line. The main feature is that it is possible to eliminate all unphysical singularities of physical fields, e.g., dislocation density tensor and elastic bend-twist tensor which are still singular in the first strain gradient elasticity.

Keywords: Gradient elasticity; Nonlocal elasticity; Dislocations; Double stress; Triple stress

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1. Introduction

Gradient elasticity theories are generalizations of linear elasticity which include higher-order terms to account for microstructural or couple stress effects in materials. Strain gradient theories were introduced by Kröner (1963), Kröner and Datta (1966), Kröner (1967), Mindlin (1964, 1965), Mindlin and Eshel (1968), Green and Rivlin (1964a,b) in the sixties. In a strain gradient theory the strain energy depends on the elastic strain and gradients of the elastic strain. Due to the gradients, such theories contain additional coefficients with the dimension of a length which are called gradient coefficients. In addition to Cauchy-like stress tensors, hyperstresses (e.g. double stresses and triple stresses) occur in such a framework. But most of all applications used a first strain gradient theory instead of a second strain gradient elasticity. One reason is that the second-order strain gradient theory is mathematically more involved and first strain gradient theories are more simple to handle. A special version of Mindlin's first strain gradient theory with only one gradient coefficient can be successfully employed to calculate the elastic fields of cracks, dislocations and disclinations. Gradient elasticity was used to calculate the stress and the strain fields produced by dislocations and disclinations (Gutkin and Aifantis, 1999; Gutkin, 2000; Aifantis, 2003; Lazar and Maugin, in press). The gradient elasticity solutions have no singularities in both the stress and the strain fields. On the other hand, in first gradient elasticity the double stresses still have singularities at the defect line (Lazar and Maugin, in press). Thus, one would expect that the double stresses are nonsingular and, on the other hand, the triple stresses are still singular in the framework of second strain gradient elasticity. But, is this true?

In addition, the stress of such a special (static) gradient elasticity may correspond to the stress in Eringen's (static) theory of nonlocal elasticity. Then, it is a one to one relationship between the stresses calculated in gradient elasticity and the stresses in nonlocal elasticity.

Recently, Eringen (1992, 2002) proposed the following equation in nonlocal isotropic elasticity:

$$[1 - \varepsilon^2 \Delta + \gamma^4 \Delta^2]\sigma_{ij} = \sigma_{ij}^{(\mathrm{cl})}, \tag{1.1}$$

where $\sigma_{ij}^{(\mathrm{cl})}$ denotes the stress in 'classical' elasticity, and ε and γ are two positive parameters of nonlocality. But this equation has not yet been used to find stresses of dislocations and disclinations. Only, the case if $\gamma = 0$ in Eq. (1.1) has been used for applications. Thus, solutions of Eq. (1.1) for dislocations, disclinations and cracks are missing. On the other hand, this author has not calculated the corresponding nonlocal kernels. Of course, the nonlocal kernel can be the Green function of (1.1). Are the nonlocal kernels singular or nonsingular? What is the form of the corresponding gradient elasticity? All these points are worth an investigation.

In the meantime, Lazar et al. (in press) have investigated Eq. (1.1) within the theory of nonlocal elasticity of bi-Helmholtz type. They found smooth nonlocal stress fields for screw and edge dislocations. Nevertheless, in nonlocal elasticity the elastic strain and the total displacement vector have the classical form (singularities and discontinuity). Thus, the elastic strain is still singular at the dislocation line. Can a gradient theory eliminate these singularities? Lazar et al. (in press) calculated the nonlocal kernel of bi-Helmholtz type and discovered that the kernel is nonsingular in one-, two- and three-dimensions. In addition, they compared the dispersion relation in such a nonlocal theory with the one obtained in models of lattice dynamics and found, in this way, certain values for the two parameters of nonlocality in terms of lattice parameters.

Some other questions arise in gradient elasticity. Is it possible to regularize all unphysical singularities which appear? On the one hand, the stresses and strains are nonsingular in a first strain gradient theory but, on the other hand, the components of the bend-twist tensor and the double stress tensor still have singularities. In gradient elasticity all higher order stresses (hyperstresses) should be nonsingular. Can we reach this goal by means of a second strain gradient elasticity or must we consider a triple or even higher gradient elasticity which would be more complex? In addition, not so much is known about triple stresses.

In the present paper, for the first time, we want to examine dislocations in a static theory of second strain gradient elasticity with double and triple stresses. We are discussing the general framework of such a gradient elasticity theory in greater detail. Such a gradient theory should be very useful for the study of dislocation core properties.

The plan of the paper is as follows. In Section 2, we derive all basic equations of second strain gradient elasticity. We give the most general anisotropic constitutive equations and simplified ones where the double stress is the first gradient of the Cauchy-like stress and the triple stress is given in terms of the second gradient of the force stress tensor. Such simplified second gradient elasticity may be connected to a nonlocal isotropic elasticity of bi-Helmholtz type as proposed by Eringen. We discuss these relations and calculate the corresponding nonlocal kernel. In Sections 3 and 4, respectively, we investigate the cases of a screw dislocation and an edge dislocation in the theory of second strain gradient elasticity in detail. We calculate the elastic stresses, strain and distortion tensors by using the stress function method. These fields have no singularities and they are slightly modified in comparison with the first strain gradient results. In addition, we calculate the double stresses, triple stresses and the dislocation density of a single screw dislocation and a single edge dislocation. We show that these fields have no singularity within the dislocation core region. Therefore, it is possible to regularize all elastic fields, which are physical state quantities, including the higher stresses within the framework of second strain gradient elasticity. In Section 5, we provide a summary. Some technical details are given in Appendices A–C.

2. Basic equations

2.1. Kinematics

In elasticity the deformation is described by a displacement vector u_i. Elasticity without defects is called compatible. If defects like dislocations or disclinations are present, one deals with incompatible elasticity.

In the classical theory of dislocations, the total distortion, denoted by β^{T}_{ij}, is given as a sum of elastic and plastic parts

$$\beta^{\mathrm{T}}_{ij} = \partial_j u_i = \beta_{ij} + \beta^{\mathrm{P}}_{ij}, \tag{2.1}$$

which is just the gradient of the displacement and, thus, a compatible distortion. The elastic (incompatible) distortion tensor is defined as (see, e.g., DeWit, 1973; Mura, 1982)

$$\beta_{ij} = \partial_j u_i - \beta^{\mathrm{P}}_{ij}. \tag{2.2}$$

Here β^{P}_{ij} denotes the plastic distortion tensor. On the other hand, the elastic distortion may be rewritten

$$\beta_{ij} = E_{ij} - \epsilon_{ijk}\omega_k, \tag{2.3}$$

where the symmetric part of (2.3) defines the (incompatible) elastic strain

$$E_{ij} = \beta_{(ij)} = \frac{1}{2}(\beta_{ij} + \beta_{ji}) \tag{2.4}$$

and the elastic rotation vector is defined by

$$\omega_k = -\frac{1}{2}\epsilon_{ijk}\beta_{ij}. \tag{2.5}$$

In the case of dislocations, the elastic bend-twist tensor is given by

$$\kappa_{ij} = \partial_j \omega_i, \tag{2.6}$$

which is just the gradient of the rotation. Therefore, it is compatible. If the plastic distortion is non-zero, the dislocation density tensor reads

$$\alpha_{ij} = \epsilon_{jkl}\partial_k \beta_{il} = -\epsilon_{jkl}\partial_k \beta^{\mathrm{P}}_{il}. \tag{2.7}$$

We see that (2.7) satisfies the continuity conditions (or Bianchi identity)

$$\partial_j \alpha_{ij} = 0. \tag{2.8}$$

The strength of a dislocation called Burgers vector is given by

$$b_i(r) = \oint_C \beta_{ij}\, \mathrm{d}x_j, \tag{2.9}$$

where C denotes the Burgers circuit around a dislocation.

Gradients of the strain tensor are called hyperstrain. The first gradient of the elastic strain is called the (elastic) double strain

$$\eta_{ijk} = \partial_k E_{ij} \tag{2.10}$$

and the triple strain is defined by

$$\eta_{ijkl} = \partial_l \partial_k E_{ij}. \tag{2.11}$$

They fulfill the following compatibility conditions:

$$\epsilon_{mlk} \partial_l \eta_{ijk} = 0, \tag{2.12}$$

$$\epsilon_{mnl} \epsilon_{pqk} \partial_n \partial_q \eta_{ijkl} = 0. \tag{2.13}$$

If the plastic strain gradient is the gradient of the plastic strain, then such model is called gradient of strain model (Forest and Sievert, 2003). However, the elastic hyperstresses may be considered as state variables in the free energy.

2.2. General case of second strain gradient elasticity

For a linear elastic solid, the potential energy function, W, is assumed to be a quadratic function in terms of strain, first-order gradient strain and second-order gradient strain

$$W = W(E_{ij}, \partial_k E_{ij}, \partial_l \partial_k E_{ij}). \tag{2.14}$$

Since the strain E_{ij} is incompatible, we deal with an incompatible strain gradient elasticity which is valid for defects (dislocations, disclinations) in linear elasticity. Then in this gradient elasticity

$$\sigma_{ij} := \frac{\partial W}{\partial E_{ij}}, \quad \sigma_{ij} = \sigma_{ji}, \tag{2.15}$$

$$\tau_{ijk} := \frac{\partial W}{\partial(\partial_k E_{ij})}, \quad \tau_{ijk} = \tau_{jik}, \tag{2.16}$$

$$\tau_{ijkl} := \frac{\partial W}{\partial(\partial_l \partial_k E_{ij})}, \quad \tau_{ijkl} = \tau_{jikl}, \quad \tau_{ijkl} = \tau_{ijlk} \tag{2.17}$$

are the response quantities with respect to E_{ij}, $\partial_k E_{ij}$ and $\partial_l \partial_k E_{ij}$. τ_{ijk} and τ_{ijkl} can be interpreted as field momenta which are canonically conjugated to the double and triple strains, respectively. Here $\sigma_{(ij)}$ possesses 6 independent components, $\tau_{(ij)k}$ has $18 = 6 \times 3 = 10 + 8$ independent components, and $\tau_{(ij)(kl)}$ possesses $36 = 6 \times 6 = 15 + 15 + 6$ independent components.[1] σ_{ij} is a Cauchy-like stress tensor, whereas τ_{ijk} and τ_{ijkl}

[1] We notice that our $\tau_{(ij)(kl)}$ is slightly different from Mindlin's triple stress tensor $\tau_{i(jkl)}$ due to $\tau_{i(jkl)} := (\partial W)/(\partial_l \partial_k \partial_j u_i)$ and $W = W(\partial_{(j} u_{i)}, \partial_k \partial_j u_i, \partial_l \partial_k \partial_j u_i)$. Thus, $\tau_{i(jkl)}$ possesses $30 = 15 + 15$ independent components. In Mindlin's gradient theory, the strain and the distortion are compatible. Therefore, one may call such a gradient theory—a compatible strain gradient theory. Such a compatible gradient theory is obtained from (2.14) when the plastic strain is zero, $E^{\mathrm{P}}_{ij} = 0$, such that $E_{ij} \equiv E^{\mathrm{T}}_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$. Then the number of independent components for the triple stress reduces to 30.

are higher order stress tensors (hyperstresses). Sometimes, τ_{ijk} and τ_{ijkl} are called double and triple stresses, respectively. The τ_{ijk} have the character of double forces (or force dipoles). The first index of τ_{ijk} describes the orientation of the pair of (antiparallel) forces F_i, the second index gives the orientation of the lever arm, Δx_j, between the forces and the third index denotes the orientation of the surface on which the forces act. On the other hand, the τ_{ijkl} have the character of triple forces (or force quadrupole) per unit area. The quadrupole of forces is a dipole of force dipoles, i.e., a dipole of two moments. The last index of τ_{ijkl} describes the orientation of the axis of the dipoles of the two moments. The other indices of τ_{ijkl} have the same meaning as for τ_{ijk}.

With Eq. (2.2) the strain energy (2.14) may be written in terms of gradients of the displacement and the plastic strain according to

$$W = W\Big(\partial_{(j}u_{i)}, \partial_k\partial_{(j}u_{i)}, \partial_k\partial_l\partial_{(j}u_{i)}, E^{\mathrm{P}}_{ij}, \partial_k E^{\mathrm{P}}_{ij}, \partial_l\partial_k E^{\mathrm{P}}_{ij}\Big), \tag{2.18}$$

where $E^{\mathrm{P}}_{ij} = \beta^{\mathrm{P}}_{(ij)}$ denotes the plastic strain.

The force equilibrium condition follows from the variation of W with respect to the displacement vector u_i:

$$\partial_j(\sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}) = 0. \tag{2.19}$$

We do not give here the associated boundary conditions because we consider an infinitely extended medium. The interested reader can find boundary conditions derived in second strain elasticity by Mindlin (1965), Jaunzemis (1967), Wu (1992), Polizzotto (2003). If we define the total stress tensor

$$\overset{\circ}{\sigma}_{ij} = \sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}, \tag{2.20}$$

Eq. (2.19) takes the form

$$\partial_j\overset{\circ}{\sigma}_{ij} = 0. \tag{2.21}$$

When we add an additional 'Lagrangian', which in the compatible case (no plastic distortion) is a null Lagrangian

$$W' = -\overset{\circ}{\sigma}_{ij}E_{ij}, \tag{2.22}$$

to W, we can obtain Eq. (2.20) as variation with respect to the plastic strain E^{P}_{ij}.

For anisotropic elasticity, W may have the form

$$\begin{aligned} W = {} & \frac{1}{2}C_{ijkl}E_{ij}E_{kl} + \frac{1}{2}C_{ijklmn}(\partial_k E_{ij})(\partial_n E_{lm}) + \frac{1}{2}C_{ijklmnpq}(\partial_l\partial_k E_{ij})(\partial_q\partial_p E_{mn}) + D_{ijklm}E_{ij}\partial_m E_{kl} \\ & + D_{ijklmn}E_{ij}(\partial_n\partial_m E_{kl}) + D_{ijklmnp}(\partial_k E_{ij})(\partial_p\partial_n E_{lm}), \end{aligned} \tag{2.23}$$

where the last three contributions are cross terms. From Eq. (2.23) we obtain the constitutive equations:

$$\sigma_{ij} = C_{ijkl}E_{kl} + D_{ijklm}\partial_m E_{kl} + D_{ijklmn}\partial_n\partial_m E_{kl}, \tag{2.24}$$

$$\tau_{ijk} = D_{lmijk}E_{lm} + C_{ijklmn}\partial_n E_{lm} + D_{ijklmnp}\partial_p\partial_n E_{lm}, \tag{2.25}$$

$$\tau_{ijkl} = D_{mnijkl}E_{mn} + D_{mnpijkl}\partial_p E_{mn} + C_{ijklmnpq}\partial_q\partial_p E_{mn}, \tag{2.26}$$

which agree with the constitutive relations given earlier by Kröner and Datta (1966). Here C_{ijkl}, C_{ijklmn}, $C_{ijklmnpq}$, D_{ijklm}, D_{ijklmn} and $D_{ijklmnp}$ are constitutive coefficients, which satisfy the symmetry relations

$$C_{ijkl} \equiv C_{(ij)(kl)}, \quad C_{(ij)(kl)} = C_{(kl)(ij)},$$

$$C_{ijklmn} \equiv C_{(ij)k(lm)n}, \quad C_{(ij)k(lm)n} = C_{(lm)k(ij)n} = C_{(ij)n(lm)k} = C_{(lm)n(ij)k},$$

$$
\begin{aligned}
&C_{ijklmnpq} \equiv C_{(ij)(kl)(mn)(pq)}, \quad C_{(ij)(kl)(mn)(pq)} = C_{(mn)(kl)(ij)(pq)} = C_{(ij)(pq)(mn)(kl)} = C_{(mn)(pq)(ij)(kl)},\\
&D_{ijklm} \equiv D_{(ij)(kl)m}, \quad D_{(ij)(kl)m} = D_{(kl)(ij)m},\\
&D_{ijklmn} \equiv D_{(ij)(kl)(mn)}, \quad D_{(ij)(kl)(mn)} = D_{(kl)(ij)(mn)},\\
&D_{ijklmnp} \equiv D_{(ij)k(lm)(np)}, \quad D_{(ij)k(lm)(np)} = D_{(lm)k(ij)(np)} = D_{(ij)n(lm)(kp)} = D_{(ij)p(lm)(kn)}.
\end{aligned}
\tag{2.27}
$$

We notice that the exact meaning of a Cauchy stress is blurred and the direct connection between stress and strain of the same order is lost. Thus, instead of the Hooke law $\sigma_{ij} = C_{ijkl}E_{kl}$ the more complicated relation (2.24) is valid.

2.3. Exceptional case of second strain gradient elasticity

In order to simplify the higher gradient elasticity and to connect it with the nonlocal isotropic elasticity proposed by Eringen (1992, 2002), all crossing terms D_{ijklm}, D_{ijklmn} and $D_{ijklmnp}$ must be zero and the higher order stress tensors are just simple gradients of the Cauchy-like stress tensor multiplied by two gradient coefficients:

$$\sigma_{ij} = C_{ijkl}E_{kl}, \tag{2.28}$$

$$\tau_{ijk} = \varepsilon^2 C_{ijmn}\partial_k E_{mn} = \varepsilon^2 \partial_k \sigma_{ij}, \tag{2.29}$$

$$\tau_{ijkl} = \gamma^4 C_{ijmn}\partial_l \partial_k E_{mn} = \gamma^4 \partial_l \partial_k \sigma_{ij}. \tag{2.30}$$

Both ε and γ are gradient coefficients with the dimension of a length.

Then in such a particular second gradient model, W in (2.14) has the following simple form:

$$W = \frac{1}{2}\sigma_{ij}E_{ij} + \frac{1}{2}\varepsilon^2(\partial_k\sigma_{ij})(\partial_k E_{ij}) + \frac{1}{2}\gamma^4(\partial_l\partial_k\sigma_{ij})(\partial_l\partial_k E_{ij}), \tag{2.31}$$

which has been proposed by Lazar and Maugin (in press), Polizzotto (2003). It contains, in particular, the tensor of elastic moduli and two gradient coefficients, only. It is important to note that the energy (2.31) is valid for elastic media with double and triple stresses which are simple gradients of the force stress, a rather peculiar case, we admit. The first contribution in Eq. (2.31) has the same form as in elasticity and the second and third contributions are the gradient terms which appear in the theory of higher order gradient elasticity. In the isotropic case, the tensor of elastic moduli reads

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}), \tag{2.32}$$

where λ and μ are the Lamé constants. It is important to note that it is not possible to get the constitutive relation (2.30) together with (2.32) by setting some material parameters in Mindlin's theory (Mindlin, 1965) of second gradient of strain to be zero. The reason why is that he used the symmetry $\tau_{i(ijk)}$ instead of $\tau_{(ij)(kl)}$ in the isotropic constitutive equation for the triple stress. But, of course, the double stress (2.29) with (2.32) can be obtained by setting some material coefficients to be zero (see, e.g., Lazar and Maugin, in press). Thus, we point out that the constitutive relation for the triple stresses is a difference to Mindlin's theory. However, the formal form of the field equations is the same.

Now combining Eqs. (2.29) and (2.30) with (2.19), we obtain

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)\partial_j\sigma_{ij} = 0, \tag{2.33}$$

where Δ is the Laplacian and $\Delta\Delta$ is the bi-Laplacian. Using Eqs. (2.20), (2.29), and (2.30), we obtain the following inhomogeneous partial differential equation (PDE) of fourth-order for the Cauchy-like stress:

$$\left(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta\right)\sigma_{ij} = \overset{\circ}{\sigma}_{ij}, \tag{2.34}$$

where the inhomogeneous part is given by the total stress tensor. If we compare Eq. (1.1) and (2.34), the total stress tensor $\overset{\circ}{\sigma}_{ij}$ may be identified with the 'classical' stress tensor $\sigma_{ij}^{(\mathrm{cl})}$. Then, Eq. (2.34) has the same form as the PDE of fourth-order (1.1) proposed by Eringen in nonlocal elasticity (Eringen, 1992, 2002). σ_{ij} is a modified stress due to the Laplacian terms in (2.34). The gradient coefficients or parameters of nonlocality may be expressed in a more appropriate form as

$$\varepsilon = e_0 a, \quad \gamma = \gamma_0 a, \tag{2.35}$$

where a is an internal characteristic length (e.g., lattice parameter, granular distance), and e_0 and γ_0 are constants appropriate to each material. Thus, the stresses of this higher gradient elasticity must be equal to the stresses in Eringen's nonlocal elasticity. In Lazar et al. (in press), the coefficients ε_0 and γ_0 are determined from dispersion relations in nonlocal elasticity of bi-Helmholtz type and their matching with lattice models.

Alternatively, the PDE of fourth-order (2.34) may be decomposed into a product of two differential operators of second-order of Helmholtz-type as follows:

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta)\sigma_{ij} = \overset{\circ}{\sigma}_{ij}, \tag{2.36}$$

where we introduced the auxiliary parameters

$$c_1^2 = \frac{\varepsilon^2}{2}\left(1 + \sqrt{1 - 4\frac{\gamma^4}{\varepsilon^4}}\right), \tag{2.37}$$

$$c_2^2 = \frac{\varepsilon^2}{2}\left(1 - \sqrt{1 - 4\frac{\gamma^4}{\varepsilon^4}}\right) \tag{2.38}$$

and

$$\varepsilon^2 = c_1^2 + c_2^2, \tag{2.39}$$

$$\gamma^4 = c_1^2 c_2^2. \tag{2.40}$$

For this reason, Eq. (2.36) is called bi-Helmholtz-equation. It can be seen that these two coefficients are real, by examining the discriminant:

$$0 \leqslant \left(1 - 4\frac{\gamma^4}{\varepsilon^4}\right), \tag{2.41}$$

which is necessary to fulfill the condition (2.40). For the two coefficients it holds:

- $\varepsilon^4 > 4\gamma^4$, $c_1 \neq c_2$ are real, $\Rightarrow \varepsilon > \sqrt{2}\gamma$
- $\varepsilon^4 = 4\gamma^4$, $c_1 = c_2$ are real, $\Rightarrow \varepsilon = \sqrt{2}\gamma$.

Thus, both coefficients c_1 and c_2 have to be real. In the second case, we can reduce the two coefficients to only one.

The first-order gradient elasticity is obtained from the second-order in the limit $\gamma \to 0$. So, we get $c_1^2 \to \varepsilon^2$ and $c_2^2 \to 0$. In addition, the conditions $3\lambda + 2\mu > 0$, $\mu > 0$ and $\varepsilon^2 > 0$ were proven by Georgiadis et al. (2004) in order to have stability for the field equation of first strain gradient elasticity. Thus, ε and c_1 are real and not complex in this limit. First-order results can be obtained from the second-order results in this limit.

The solution of Eq. (2.34) may be rewritten as a convolution integral

$$\sigma_{ij}(r) = \int_V G(r - r')\overset{\circ}{\sigma}_{ij}(r')\,\mathrm{d}v(r'), \tag{2.42}$$

where $G(r)$ denotes the Green function which may be identified with the nonlocal kernel. Therefore, it has to satisfy

$$\left(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta\right)G(r)=\delta(x)\delta(y) \tag{2.43}$$

and

$$\left(1-c_1^2\Delta\right)\left(1-c_2^2\Delta\right)G(r)=\delta(x)\delta(y) \tag{2.44}$$

respectively. Eqs. (2.43) and (2.44) have two-parameter solutions whose behaviour at infinity is dominated by exponential decay.

For two-dimensional problems, the nonlocal kernel is given by (Appendix A)

$$G(r)=\frac{1}{2\pi}\frac{1}{c_1^2-c_2^2}\left[K_0(r/c_1)-K_0(r/c_2)\right] \tag{2.45}$$

and for $c_1 \to c_2 = \gamma$:

$$G(r)=\frac{1}{2\pi}\frac{r}{2\gamma^3}K_1(r/\gamma), \tag{2.46}$$

where K_n denotes the modified Bessel function of the second kind and n is the order of this function. It is important to note that the new nonlocal kernels (2.45) and (2.46) are nonsingular ones in contrast to the two-dimensional nonlocal kernel of the Helmholtz equation, $G(r)=1/[2\pi\varepsilon^2]K_0(r/\varepsilon)$, which also appears in first strain gradient elasticity (see, e.g., Lazar and Maugin, in press). In fact, at $r=0$ the nonlocal kernels (2.45) and (2.46) have the following maximum values:

$$G(0)=\frac{1}{2\pi}\frac{1}{c_1^2-c_2^2}\ln\frac{c_1}{c_2} \quad \text{and} \quad G(0)=\frac{\gamma}{4\pi}, \tag{2.47}$$

respectively.

In the k-space the nonlocal kernel or Green's function corresponding to Eq. (2.43) is the inverse of a polynomial of fourth degree

$$\overline{G}(k)=\left(1+\varepsilon^2k^2+\gamma^4k^4\right)^{-1}, \tag{2.48}$$

which was originally proposed by Kunin (1983) for the Debye quasicontinuum and also used by Eringen (1992, 2002) in nonlocal elasticity. But corresponding expressions in the r-space are missing. The nonlocal kernels of bi-Helmholtz type and the related nonlocal elasticity are discussed more in detail by Lazar et al. (in press).

Notice that, if we replace $-c_1^2$ by $+c_1^2$ and/or $-c_2^2$ by $+c_2^2$ in Eqs. (2.36) and (2.44), then c_1 and/or c_2 would be complex and the solutions would be given in terms of the Hankel function $\frac{i\pi}{2}H_0^{(1)}(r/c_1)$ and/or $\frac{i\pi}{2}H_0^{(1)}(r/c_2)$ instead of the modified Bessel function $K_0(r/c_1)$ and/or $K_0(r/c_2)$. The nonlocal kernel would possess an oscillatory character even in the far field (Eringen, 1987) and would not be short-ranged. On the other hand, the solutions for the stresses σ_{ij} obtained from the inhomogeneous bi-Helmholtz equation (2.36) would be modified in the far field. The decay of the stresses and strains would be oscillatory.

Using the inverse of the Hooke law with the same material constants for σ_{ij} and $\overset{\circ}{\sigma}_{ij}$ we obtain from Eq. (2.34) the PDE of fourth-order for the elastic strain

$$\left(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta\right)E_{ij}=\overset{\circ}{E}_{ij}, \tag{2.49}$$

where $\overset{\circ}{E}_{ij}$ denotes the elastic strain tensor calculated in classical elasticity.

If we use the decomposition (2.3), we obtain the coupled partial differential equation

$$\left(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta\right)\left[\partial_{(i}u_{j)}-\beta^{\mathrm{P}}_{(ij)}\right]=\partial_{(i}\overset{\circ}{u}_{j)}-\overset{\circ}{\beta}{}^{\mathrm{P}}_{(ij)}, \tag{2.50}$$

where $\overset{\circ}{u}_i$ denotes the displacement field and $\overset{\circ}{\beta}{}^{\mathrm{P}}_{ij}$ is the plastic distortion in classical defect theory (see, e.g., DeWit, 1973; Mura, 1982). Thus, if the following equations are fulfilled:

$$\left(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta\right)\beta_{ij}=\overset{\circ}{\beta}_{ij}, \tag{2.51}$$

$$\left(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta\right)\beta^{\mathrm{P}}_{ij}=\overset{\circ}{\beta}{}^{\mathrm{P}}_{ij}, \tag{2.52}$$

the equation for the displacement field,[2]

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)u_i=\overset{\circ}{u}_i, \tag{2.53}$$

is valid for the incompatible case. In the classical theory of defects (dislocations, disclinations) the plastic distortion and the total displacement are discontinuous fields. Thus, one must solve the Eqs. (2.52) and (2.53) with discontinuities as inhomogeneous parts.

We notice that in second strain gradient elasticity the following inhomogeneous PDE is valid for the dislocation density tensor:

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)\alpha_{ij}=\overset{\circ}{\alpha}_{ij}. \tag{2.54}$$

Eq. (2.54) is calculated as the curl of (2.51). In second gradient elasticity, the dislocation density tensor of a single and straight dislocation is given by

$$\alpha_{ij}=b_i\otimes n_jG(r), \tag{2.55}$$

where $G(r)$ is the nonlocal kernel (2.45) or (2.46), whereas the dislocation density in classical elasticity reads

$$\overset{\circ}{\alpha}_{ij}=b_i\otimes n_j\delta(x)\delta(y), \tag{2.56}$$

where n_j denotes the direction of the dislocation line.

In order to use the stress function method, the stress σ_{ij} should fulfill

$$\partial_j\sigma_{ij}=0. \tag{2.57}$$

Using Eq. (2.57), we obtain from (2.19)

$$\partial_j\partial_k\tau_{ijk}=0,\quad \partial_l\partial_k\partial_j\tau_{ijkl}=0. \tag{2.58}$$

It is obvious that (2.58) is satisfied by Eq. (2.57) and the constitutive relations (2.29) and (2.30). We note that the relation (2.57) is a constraint which specifies the structure of the solution for σ_{ij}. By the help of such a constraint we will be able to introduce modified Prandtl and Airy stress functions for the stress in gradient elasticity.

3. Screw dislocation

In this section, we consider a straight screw dislocation within the theory of second gradient elasticity. The dislocation is situated in an infinitely extended body. The dislocation line and the Burgers vector of the screw dislocation coincide with the z-axis.

[2] If $\beta^{\mathrm{P}}_{ij}=0$ (compatible distortion), the inhomogeneous Helmholtz equation, $(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)u_i=\overset{\circ}{u}_i$, is obtained without further assumptions.

3.1. Solution in classical elasticity

The ‘classical’ expressions of the force stress of a screw dislocation have traditionally been calculated by using the theory of linear elasticity. The ‘classical’ stress function is given by

$$\overset{\circ}{F} = \frac{\mu b_z}{2\pi} \ln r \tag{3.1}$$

and $r = \sqrt{x^2 + y^2}$. Sometimes, the stress function (3.1) is called the Prandtl stress function. The force stress is given in terms of the stress function (3.1)

$$\overset{\circ}{\sigma}_{zx} = -\partial_y \overset{\circ}{F} = -\frac{\mu b_z}{2\pi} \frac{y}{r^2}, \quad \overset{\circ}{\sigma}_{zy} = \partial_x \overset{\circ}{F} = \frac{\mu b_z}{2\pi} \frac{x}{r^2}. \tag{3.2}$$

It has a nasty $1/r$-singularity at the dislocation line.

3.2. Solution in second strain gradient elasticity

We make the following stress function ansatz for the stress tensor:

$$\sigma_{zx} = -\partial_y F, \quad \sigma_{zy} = \partial_x F, \tag{3.3}$$

which fulfills Eq. (2.57). It has the same form as the stress function ansatz for the classical stress tensor (3.2). Here F denotes the modified stress function which must be determined. If we substitute (3.3) and (3.2) into the bi-Helmholtz equation for the stress tensor (2.34), we obtain for the modified stress function the following inhomogeneous PDE of fourth-order:

$$\left(1 - \varepsilon^2 \Delta + \gamma^4 \Delta\Delta\right) F = \frac{\mu b_z}{2\pi} \ln r, \tag{3.4}$$

where the inhomogeneous part is given by the stress function (3.1). Alternatively, we obtain from the factorized PDE (2.36) a bi-Helmholtz equation for the stress function

$$\left(1 - c_1^2 \Delta\right)\left(1 - c_2^2 \Delta\right) F = \frac{\mu b_z}{2\pi} \ln r. \tag{3.5}$$

The solution of (3.5) is given by (Appendix B)

$$F = \frac{\mu b_z}{2\pi} \left\{ \ln r + \frac{1}{c_1^2 - c_2^2} \left[c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2) \right] \right\}. \tag{3.6}$$

In the limit $c_1 \to c_2$, Eq. (3.6) simplifies to

$$F = \frac{\mu b_z}{2\pi} \left\{ \ln r + K_0(r/\gamma) + \frac{r}{2\gamma} K_1(r/\gamma) \right\}. \tag{3.7}$$

Because of the bi-Helmholtz equation (3.5), one may call the stress functions (3.6) and (3.7)—the bi-Helmholtz modified Prandtl stress functions.

Using Eqs. (3.3) and (3.6), the stress tensor reads in Cartesian coordinates

$$\sigma_{zx} = -\frac{\mu b_z}{2\pi} \frac{y}{r^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2} \left[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2) \right] \right\}, \tag{3.8}$$

$$\sigma_{zy} = \frac{\mu b_z}{2\pi} \frac{x}{r^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2} \left[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2) \right] \right\} \tag{3.9}$$

and for $c_1 \to c_2$

$$\sigma_{zx} = -\frac{\mu b_z}{2\pi}\frac{y}{r^2}\left\{1 - \frac{r}{\gamma}K_1(r/\gamma) - \frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}, \tag{3.10}$$

$$\sigma_{zy} = \frac{\mu b_z}{2\pi}\frac{x}{r^2}\left\{1 - \frac{r}{\gamma}K_1(r/\gamma) - \frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}. \tag{3.11}$$

They are zero at $r = 0$ and have extremum values near the dislocation line. The extremum values depend strongly on c_2 and c_1. For $c_1 = c_2 = \gamma$, we have: $|\sigma_{zx}(0,y)| \simeq 0.249\mu b_z/[2\pi\gamma] = 0.352\mu b_z/[2\pi\varepsilon]$ at $|y| \simeq 2.324\gamma = 1.643\varepsilon$ and $|\sigma_{zy}(x,0)| \simeq 0.249\mu b_z/[2\pi\gamma] = 0.352\mu b_z/[2\pi\varepsilon]$ at $|x| \simeq 2.324\gamma = 1.643\varepsilon$. Eqs. (3.10) and (3.11) are plotted in Fig. 1. In the limits $c_2 \to 0$ and $c_1 \to \varepsilon$, the stresses which are calculated in first strain gradient elasticity (Gutkin and Aifantis, 1999; Gutkin, 2000; Lazar, 2003b) or in the corresponding nonlocal elasticity (Eringen, 1983, 2002) are recovered. The stresses have in first strain elasticity the following maxima: $|\sigma_{zx}(0,y)| \simeq 0.399\mu b_z/[2\pi\varepsilon]$ at $|y| \simeq 1.114\varepsilon$ and $|\sigma_{zy}(x,0)| \simeq 0.399\mu b_z/[2\pi\varepsilon]$ at $|x| \simeq 1.114\varepsilon$. Only in the region $r/\varepsilon < 3$ is a difference between the second-order and the first-order stresses (see Fig. 1). We

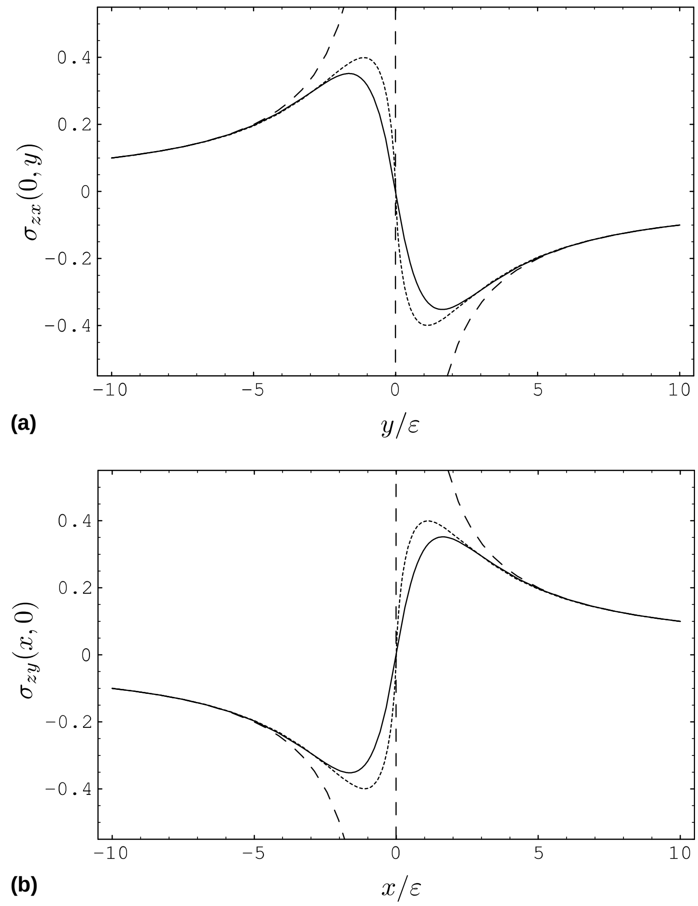

Fig. 1. Stresses of a screw dislocation: (a) $\sigma_{zx}(0,y)$ and (b) $\sigma_{zy}(x,0)$ (full curves) are given in units of $\mu b_z/[2\pi\varepsilon]$. The full curves, small dashed curves and dashed curves, respectively, represent the stress fields in gradient elasticity of bi-Helmholtz type, gradient elasticity of Helmholtz type and classical elasticity.

notice that the stresses (3.8)–(3.11) agree with the solutions in the nonlocal elasticity of bi-Helmholtz type-given by Lazar et al. (in press).

Using the inverse of the Hooke law (2.28), the elastic strain tensor reads

$$E_{zx} = -\frac{b_z}{4\pi}\frac{y}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right\}, \tag{3.12}$$

$$E_{zy} = \frac{b_z}{4\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right\} \tag{3.13}$$

and with $c_1 = c_2 = \gamma$

$$E_{zx} = -\frac{b_z}{4\pi}\frac{y}{r^2}\left\{1 - \frac{r}{\gamma}K_1(r/\gamma) - \frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}, \tag{3.14}$$

$$E_{zy} = \frac{b_z}{4\pi}\frac{x}{r^2}\left\{1 - \frac{r}{\gamma}K_1(r/\gamma) - \frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}. \tag{3.15}$$

Again, they are zero at $r = 0$ and have extremum values near the dislocation line. The extremum values depend on c_2 and c_1. For instance, with $c_1 = c_2$ we have: $|E_{zx}(0, y)| \simeq 0.249 b_z/[4\pi\gamma] = 0.352\mu b_z/[2\pi\varepsilon]$ at $|y| \simeq 2.324\gamma = 1.643\varepsilon$ and $|E_{zy}(x, 0)| \simeq 0.249 b_z/[4\pi\gamma] = 0.352\mu b_z/[2\pi\varepsilon]$ at $|x| \simeq 2.324\,\gamma = 1.643\varepsilon$. The components of stress and strain have no singularities. They are zero at $r = 0$ and have extremum values near the dislocation line. Also, it is interesting to notice that the strains (3.12) and (3.13) have the same form as the micropolar distortions γ_{xz} and γ_{yz} produced by a screw dislocation in gradient micropolar elasticity (see Lazar and Maugin, 2004a), if we substitute $c_2 \to 1/\kappa$ and $c_1 \to 1/\tau$. In the limit $c_2 \to 0$, we recover in Eqs. (3.12) and (3.13) the strain components calculated in first strain gradient elasticity of Helmholtz type (Gutkin and Aifantis, 1996, 1999; Lazar and Maugin, in press).

Now we calculate the elastic distortion and dislocation density tensors of the screw dislocation. From the conditions that the following components of the dislocation density tensor must vanish:

$$\alpha_{xy} = -\partial_x(E_{xz} + \omega_y) \equiv 0, \quad \alpha_{yx} = \partial_y(E_{yz} - \omega_x) \equiv 0 \tag{3.16}$$

and for the elastic distortion

$$\beta_{xz} \equiv 0, \quad \beta_{yz} \equiv 0, \tag{3.17}$$

we obtain for the non-vanishing components of the rotation vector

$$\omega_x = E_{yz}, \quad \omega_y = -E_{xz}. \tag{3.18}$$

Consequently, the non-vanishing components of the elastic distortion are given by

$$\beta_{zx} = 2E_{zx}, \quad \beta_{zy} = 2E_{zy}. \tag{3.19}$$

Using (2.9) and (3.19), the effective Burgers vector can be calculated as

$$b_z(r) = \oint_C (\beta_{zx}\,\mathrm{d}x + \beta_{zy}\,\mathrm{d}y) = b_z\left\{1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right\}. \tag{3.20}$$

It depends on the radius r and the coefficients c_1 and c_2. For $c_1 = c_2 = \gamma$, the Burgers vector reads

$$b_z(r) = b_z\left\{1 - \frac{r}{\gamma}K_1(r/\gamma) - \frac{r^2}{2\gamma^2}K_0(r/\gamma)\right\}. \tag{3.21}$$

Eq. (3.21) is plotted in Fig. 2. In fact, we find $b_z(0) = 0$ and $b_z(\infty) = b$. This effective Burgers vector differs appreciably from the constant value b in the core region from $r = 0$ up to $r \simeq 6\varepsilon$ (see Fig. 2). Therefore, the core radius is given in quite a natural manner within gradient elasticity. Outside this core region the Burgers

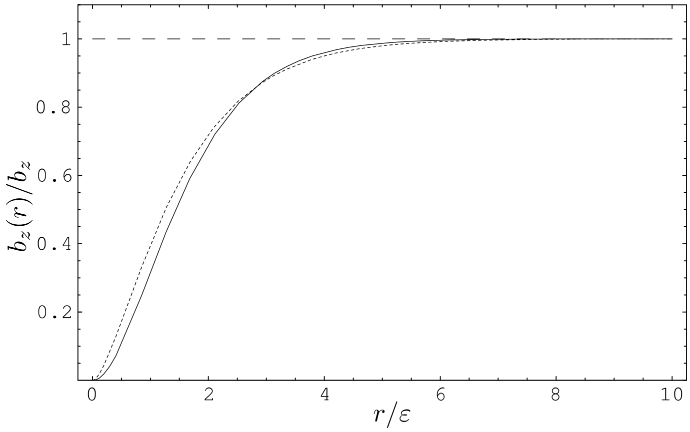

Fig. 2. The effective Burgers vector for a screw dislocation: $b_z(r)/b_z$ (full curve) and $b_z^{(1)}(r)/b_z$ (small dashed curve). The dashed curve represents the classical component.

vector reaches its constant value. In the limits $c_2 \to 0$ and $c_1 \to \varepsilon$, we obtain in Eq. (3.20) the Burgers vector calculated in first strain gradient elasticity (Lazar, 2003b)

$$b_z^{(1)}(r) = b_z\left\{1 - \frac{r}{\varepsilon}K_1(r/\varepsilon)\right\}. \tag{3.22}$$

In Fig. 2, it can be seen that the difference between (3.21) and (3.22) is small.

For the dislocation density tensor of the screw dislocation we calculate

$$\alpha_{zz} = \frac{1}{\mu}\Delta F = \frac{b_z}{2\pi}\frac{1}{c_1^2 - c_2^2}[K_0(r/c_1) - K_0(r/c_2)]. \tag{3.23}$$

At $r = 0$ the dislocation density of a single screw dislocation (3.23) has the maximum value

$$\alpha_{zz} = \frac{b_z}{2\pi}\frac{1}{c_1^2 - c_2^2}\ln\frac{c_1}{c_2}. \tag{3.24}$$

Thus, it is nonsingular. For $c_1 = c_2 = \gamma$ we obtain

$$\alpha_{zz} = \frac{b_z}{2\pi}\frac{r}{2\gamma^3}K_1(r/\gamma). \tag{3.25}$$

At $r = 0$ the dislocation density of a single screw dislocation (3.25) has the maximum value

$$\alpha_{zz} = \frac{b_z}{2\pi\varepsilon^2}. \tag{3.26}$$

Therefore, it is nonsingular unlike the dislocation density of a screw dislocation calculated in first strain gradient elasticity ($c_2 \to 0$ and $c_1 \to \varepsilon$)

$$\alpha_{zz}^{(1)} = \frac{b_z}{2\pi\varepsilon^2}K_0(r/\varepsilon), \tag{3.27}$$

which is singular at $r = 0$. Eqs. (3.25) and (3.27) are plotted in Fig. 3. It is important to note that the elimination of the singularity at $r = 0$ is a new important feature of second strain gradient elasticity.

3.3. Higher order stresses

In this section, we calculate the higher order stresses, like double and triple stresses, produced by a screw dislocation.

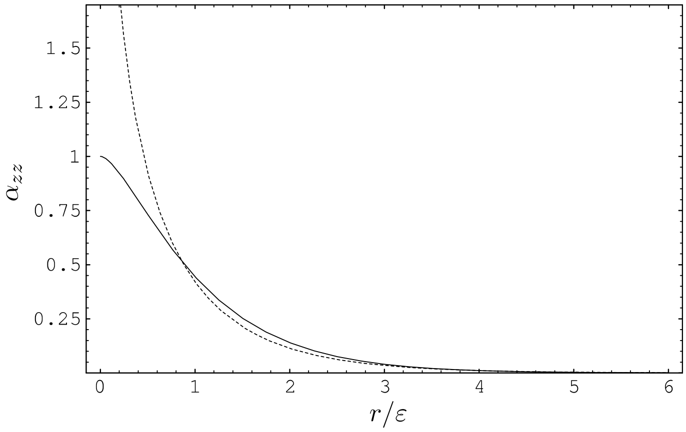

Fig. 3. Dislocation density of a single screw dislocation: α_{zz} (full curve) and $\alpha_{zz}^{(1)}$ (small dashed curve) are given in units of $b_z/[2\pi\varepsilon^2]$.

3.3.1. Double stresses

The double stresses are given by

$$\tau_{(zx)y} = -\varepsilon^2 \partial^2_{yy} F, \quad \tau_{(zy)x} = \varepsilon^2 \partial^2_{xx} F, \quad \tau_{(zx)x} = -\tau_{(zy)y} = -\varepsilon^2 \partial^2_{xy} F. \tag{3.28}$$

The result of the calculation is given in Cartesian coordinates as follows:

$$\begin{aligned}\tau_{(zy)x} = \frac{\mu b_z \varepsilon^2}{2\pi} \Bigg\{ & \frac{y^2 - x^2}{r^4} \left(1 - \frac{1}{c_1^2 - c_2^2} [c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right) \\ & + \frac{x^2}{r^2} \frac{1}{c_1^2 - c_2^2} [K_0(r/c_1) - K_0(r/c_2)] \Bigg\},\end{aligned} \tag{3.29}$$

$$\begin{aligned}\tau_{(zx)y} = -\frac{\mu b_z \varepsilon^2}{2\pi} \Bigg\{ & \frac{x^2 - y^2}{r^4} \left(1 - \frac{1}{c_1^2 - c_2^2} [c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right) \\ & + \frac{y^2}{r^2} \frac{1}{c_1^2 - c_2^2} [K_0(r/c_1) - K_0(r/c_2)] \Bigg\},\end{aligned} \tag{3.30}$$

$$\begin{aligned}\tau_{(zy)y} = -\frac{\mu b_z \varepsilon^2}{2\pi} \frac{xy}{r^4} \Bigg\{ & 2 \left(1 - \frac{1}{c_1^2 - c_2^2} [c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right) \\ & - r^2 \frac{1}{c_1^2 - c_2^2} [K_0(r/c_1) - K_0(r/c_2)] \Bigg\},\end{aligned} \tag{3.31}$$

and for $c_1 = c_2 = \gamma$ we obtain

$$\tau_{(zy)x} = \frac{\mu b_z \varepsilon^2}{2\pi} \left\{ \frac{y^2 - x^2}{r^4} \left(1 - \frac{r^2}{2\gamma^2} K_2(r/\gamma) \right) + \frac{x^2}{2r^2\gamma^2} \frac{r}{\gamma} K_1(r/\gamma) \right\}, \tag{3.32}$$

$$\tau_{(zx)y} = -\frac{\mu b_z \varepsilon^2}{2\pi} \left\{ \frac{x^2 - y^2}{r^4} \left(1 - \frac{r^2}{2\gamma^2} K_2(r/\gamma) \right) + \frac{y^2}{2r^2\gamma^2} \frac{r}{\gamma} K_1(r/\gamma) \right\}, \tag{3.33}$$

$$\tau_{(zy)y} = -\frac{\mu b_z \varepsilon^2}{2\pi} \frac{xy}{r^4} \left\{ 2 \left(1 - \frac{r^2}{2\gamma^2} K_2(r/\gamma) \right) - \frac{r^3}{2\gamma^3} K_1(r/\gamma) \right\}. \tag{3.34}$$

It is quite interesting to observe that the double stresses are nonsingular in second strain gradient elasticity unlike the double stresses calculated within the first strain gradient elasticity (Lazar and Maugin, in press) and first gradient micropolar elasticity (Lazar and Maugin, 2004a) which are singular at $r = 0$. In fact, in second gradient elasticity two components of the double stresses have extremum values and two components of the double stresses are zero at the dislocation line. In the case of $c_1 = c_2$, we find at $r = 0$:

$$\tau_{(zy)x} = -\tau_{(zx)y} = \frac{\mu b_z}{4\pi}, \quad \tau_{(zy)y} = -\tau_{(zx)x} = 0. \tag{3.35}$$

The components of the double stress (3.32)–(3.34) are plotted in Fig. 4. It is obvious that the main contribution of the double stresses comes from the dislocation core region. In the limit $c_2 \to 0$, Eqs. (3.29)–(3.31) convert to the double stresses in first strain gradient elasticity and, therefore, they become singular.

By means of the double stresses (3.29)–(3.31) and (3.32)–(3.34), we are able to give the expressions for the elastic bend-twist tensor. Eventually, the elastic bend-twist tensor is given by

$$\kappa_{xx} = \frac{1}{2\mu\varepsilon^2}\tau_{(zy)x}, \quad \kappa_{yy} = -\frac{1}{2\mu\varepsilon^2}\tau_{(zx)y}, \quad \kappa_{(xy)} = \frac{1}{2\mu\varepsilon^2}\tau_{(zy)y}, \quad \kappa_{kk} = \frac{1}{2}\alpha_{zz}. \tag{3.36}$$

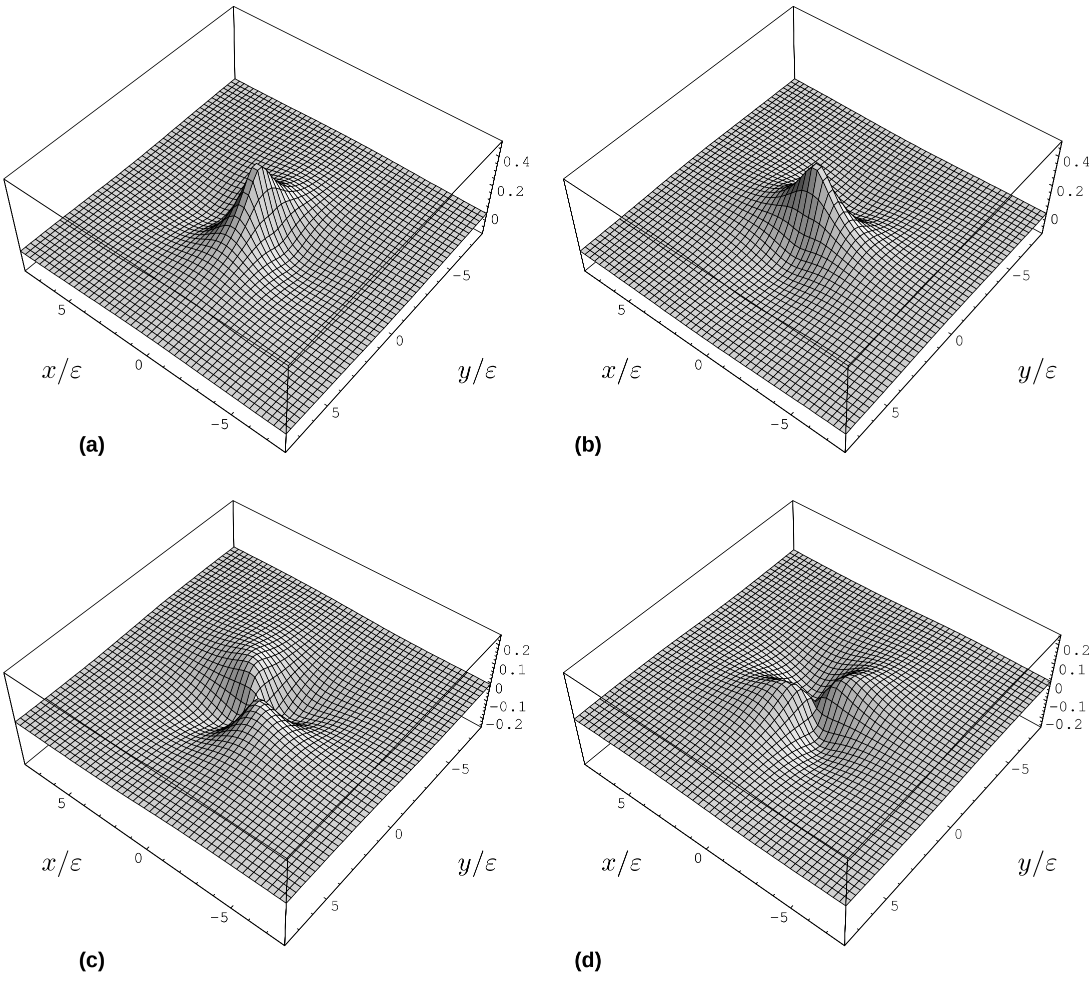

Fig. 4. Double stress of a screw dislocation: (a) $\tau_{(zy)x}$, (b) $-\tau_{(zx)y}$, (c) $\tau_{(zy)y}$ and (d) $\tau_{(zx)x}$ are given in units of $\mu b_z/[2\pi]$.

In the limits $c_2 \to 0$ and $c_1 \to 1/\kappa$, we recover the result given by Lazar (2003b). It is important to note that all singularities of the elastic bend-twist which occur in first strain gradient elasticity (Lazar, 2003b) and in micropolar elasticity (Minagawa, 1977) are removed in second strain gradient elasticity. All components of the double stress tensor and of the elastic bend-twist tensor do not have a singularity and they behave like $1/r^2$ in the far field just like an electric dipole. Thus, the double force stress may be called dipole force stress.

3.3.2. Triple stresses

For the first time ever, we want to calculate the triple stress produced by a screw dislocation. The triple stresses are given in terms of the stress function as follows:

$$\tau_{(zy)(xx)} = \gamma^4 \partial^3_{xxx} F, \quad \tau_{(zy)(xy)} = -\tau_{(zx)(xx)} = \gamma^4 \partial^3_{xxy} F, \tag{3.37}$$

$$\tau_{(zx)(yy)} = -\gamma^4 \partial^3_{yyy} F, \quad \tau_{(zy)(yy)} = -\tau_{(zx)(xy)} = \gamma^4 \partial^3_{xyy} F. \tag{3.38}$$

Eventually, we obtain

$$\tau_{(zy)(xx)} = \frac{\mu b_z \gamma^4}{2\pi} \frac{x}{r^6} \left\{ (x^2 - 3y^2) \left(2 - r^2 \frac{1}{c_1^2 - c_2^2} [K_2(r/c_1) - K_2(r/c_2)] \right) - x^2 r^2 \frac{1}{c_1^2 - c_2^2} \left[\frac{r}{c_1} K_1(r/c_1) - \frac{r}{c_2} K_1(r/c_2) \right] \right\}, \tag{3.39}$$

$$\tau_{(zx)(yy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{y}{r^6} \left\{ (y^2 - 3x^2) \left(2 - r^2 \frac{1}{c_1^2 - c_2^2} [K_2(r/c_1) - K_2(r/c_2)] \right) - y^2 r^2 \frac{1}{c_1^2 - c_2^2} \left[\frac{r}{c_1} K_1(r/c_1) - \frac{r}{c_2} K_1(r/c_2) \right] \right\}, \tag{3.40}$$

$$\tau_{(zy)(yy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{x}{r^6} \left\{ (x^2 - 3y^2) \left(2 - r^2 \frac{1}{c_1^2 - c_2^2} [K_2(r/c_1) - K_2(r/c_2)] \right) + y^2 r^2 \frac{1}{c_1^2 - c_2^2} \left[\frac{r}{c_1} K_1(r/c_1) - \frac{r}{c_2} K_1(r/c_2) \right] \right\}, \tag{3.41}$$

$$\tau_{(zy)(xy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{y}{r^6} \left\{ (y^2 - 3x^2) \left(2 - r^2 \frac{1}{c_1^2 - c_2^2} [K_2(r/c_1) - K_2(r/c_2)] \right) + x^2 r^2 \frac{1}{c_1^2 - c_2^2} \left[\frac{r}{c_1} K_1(r/c_1) - \frac{r}{c_2} K_1(r/c_2) \right] \right\} \tag{3.42}$$

and for the case of $c_1 = c_2 = \gamma$ we find

$$\tau_{(zy)(xx)} = \frac{\mu b_z \gamma^4}{2\pi} \frac{x}{r^6} \left\{ (x^2 - 3y^2) \left(2 - \frac{r^2}{\gamma^2} \left[K_2(r/\gamma) + \frac{1}{2} \frac{r}{\gamma} K_1(r/\gamma) \right] \right) - x^2 \frac{r^4}{2\gamma^4} K_0(r/\gamma) \right\}, \tag{3.43}$$

$$\tau_{(zx)(yy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{y}{r^6} \left\{ (y^2 - 3x^2) \left(2 - \frac{r^2}{\gamma^2} \left[K_2(r/\gamma) + \frac{1}{2} \frac{r}{\gamma} K_1(r/\gamma) \right] \right) - y^2 \frac{r^4}{2\gamma^4} K_0(r/\gamma) \right\}, \tag{3.44}$$

$$\tau_{(zy)(yy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{x}{r^6} \left\{ (x^2 - 3y^2) \left(2 - \frac{r^2}{\gamma^2} \left[K_2(r/\gamma) + \frac{1}{2} \frac{r}{\gamma} K_1(r/\gamma) \right] \right) + y^2 \frac{r^4}{2\gamma^4} K_0(r/\gamma) \right\}, \tag{3.45}$$

$$\tau_{(zy)(xy)} = -\frac{\mu b_z \gamma^4}{2\pi} \frac{y}{r^6} \left\{ (y^2 - 3x^2) \left(2 - \frac{r^2}{\gamma^2} \left[K_2(r/\gamma) + \frac{1}{2} \frac{r}{\gamma} K_1(r/\gamma) \right] \right) + x^2 \frac{r^4}{2\gamma^4} K_0(r/\gamma) \right\}. \tag{3.46}$$

The components (3.43)–(3.46) are plotted in Fig. 5. Thus, the triple stresses (3.39)–(3.42) and (3.43)–(3.46) are zero at the dislocation line and have extremum values near the dislocation line (see Fig. 5). These extremum values are located very close to the dislocation line. It is important to note that the triple stresses of a screw dislocation which appear in second strain gradient elasticity are nonsingular. In the limit $c_2 \to 0$,

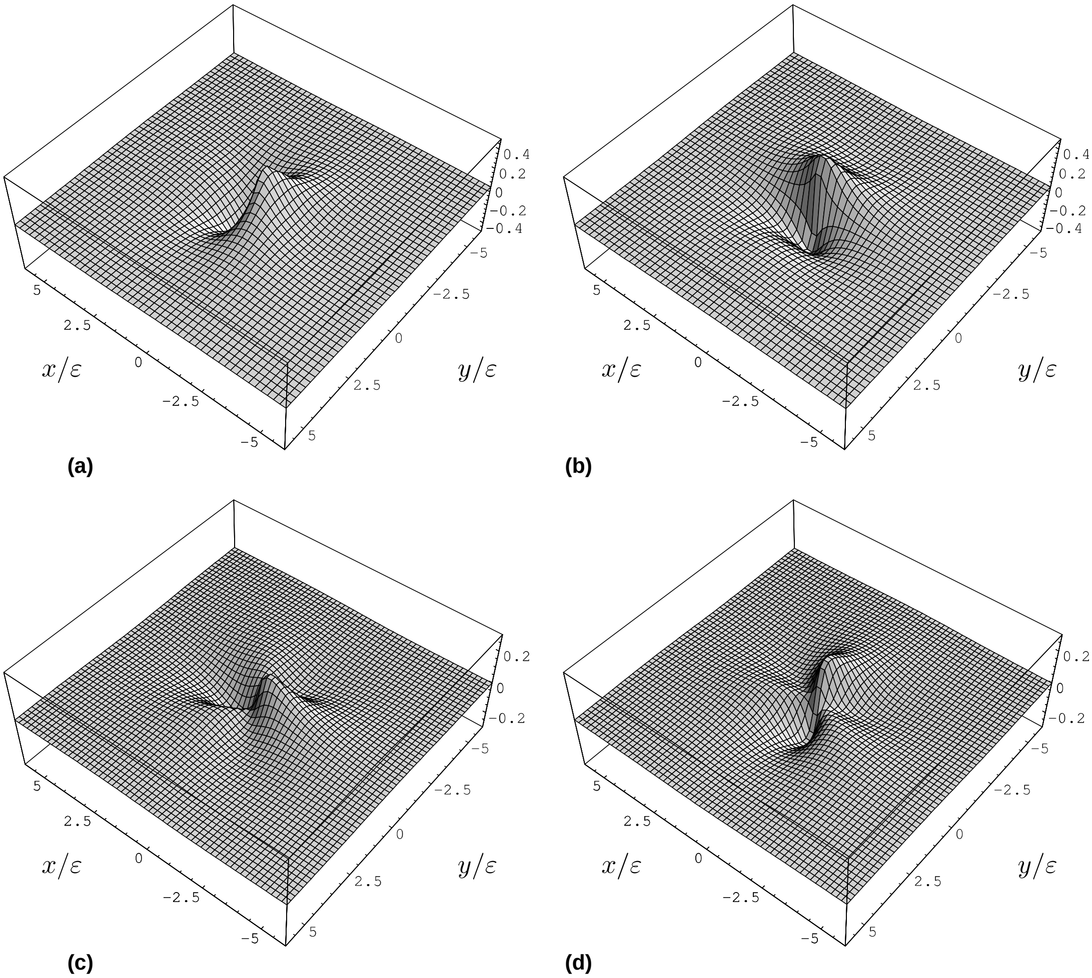

Fig. 5. Triple stress of a screw dislocation: (a) $\tau_{(zy)(xx)}$, (b) $-\tau_{(zx)(yy)}$, (c) $\tau_{(zy)(yy)}$ and (d) $\tau_{(zy)(xy)}$ are given in units of $\mu b_z\varepsilon/[8\pi]$.

Eqs. (3.39)–(3.42) become singular. In addition, they behave like $1/r^3$ in the far field just like an electric quadrupole. Therefore, the triple force stress is a quadrupole force stress.

4. Edge dislocation

In this section, we investigate a straight edge dislocation in the framework of second gradient elasticity. The dislocation line coincides with the z-axis and the Burgers vector is b_x is parallel to the x-axis.

4.1. Solution in classical elasticity

The appropriate Airy stress function for a straight edge dislocation in classical elasticity is given by

$$\chi = -\frac{\mu b_x}{2\pi(1-\nu)} y \ln r, \tag{4.1}$$

where ν is the Poisson ratio. In the case of plane strain, the stress function ansatz is given in terms of an Airy stress function as follows:

$$\overset{\circ}{\sigma}_{xx} = \partial^2_{yy}\chi, \quad \overset{\circ}{\sigma}_{yy} = \partial^2_{xx}\chi, \quad \overset{\circ}{\sigma}_{xy} = -\partial^2_{xy}\chi, \quad \overset{\circ}{\sigma}_{zz} = \nu(\overset{\circ}{\sigma}_{xx} + \overset{\circ}{\sigma}_{yy}). \tag{4.2}$$

Eventually, the ‘classical’ stresses of an edge dislocation read

$$\overset{\circ}{\sigma}_{xx} = -A\frac{y(y^2+3x^2)}{r^4}, \quad \overset{\circ}{\sigma}_{yy} = -A\frac{y(y^2-x^2)}{r^4}, \quad \overset{\circ}{\sigma}_{xy} = A\frac{x(x^2-y^2)}{r^4}, \quad \overset{\circ}{\sigma}_{zz} = -2\nu A\frac{y}{r^2}, \tag{4.3}$$

where $A = \mu b_x/[2\pi(1-\nu)]$. Thus, every component of the stress has a $1/r$-singularity at the dislocation line.

4.2. Solution in second strain gradient elasticity

For the stress in gradient elasticity we use a stress function ansatz for plane strain with the same structure as in classical elasticity

$$\sigma_{xx} = \partial^2_{yy}f, \quad \sigma_{yy} = \partial^2_{xx}f, \quad \sigma_{xy} = -\partial^2_{xy}f, \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}), \tag{4.4}$$

where f is the new stress function occurring in second gradient elasticity. The stress function ansatz (4.4) satisfies Eq. (2.57). If we use (2.34) and substituting (4.2), (4.1) and (4.4) into it, we obtain an inhomogeneous bi-Helmholtz equation for the stress function f:

$$\left(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta\right)f = -\frac{\mu b_x}{2\pi(1-\nu)}y\ln r \tag{4.5}$$

and the factorized one

$$\left(1 - c_1^2\Delta\right)\left(1 - c_2^2\Delta\right)f = -\frac{\mu b_x}{2\pi(1-\nu)}y\ln r, \tag{4.6}$$

where the inhomogeneous part is given in terms of the Airy stress function χ. The solution of this equation is given by (see Appendix C, Eq. (C.9):

$$f = -\frac{\mu b_x}{2\pi(1-\nu)}y\left\{\ln r + \frac{2(c_1^2+c_2^2)}{r^2} - \frac{2}{r(c_1^2-c_2^2)}\left[c_1^3K_1(r/c_1) - c_2^3K_1(r/c_2)\right]\right\} \tag{4.7}$$

and in the limit $c_2 \to c_1$

$$f = -\frac{\mu b_x}{2\pi(1-\nu)}y\left\{\ln r + \frac{4\gamma^2}{r^2} - \frac{4\gamma}{r}K_1(r/\gamma) - K_0(r/\gamma)\right\}. \tag{4.8}$$

Due to the bi-Helmholtz equation (4.6), one might call the stress functions (4.7) and (4.8)—the bi-Helmholtz modified Airy stress functions.

If we use the stress function (4.7), we calculate the elastic stress produced by an edge dislocation as

$$\begin{aligned}\sigma_{xx} = -\frac{\mu b_x}{2\pi(1-\nu)}\frac{y}{r^4}\bigg\{&(y^2+3x^2) + \frac{4(c_1^2+c_2^2)}{r^2}(y^2-3x^2) - \frac{2y^2}{c_1^2-c_2^2}\left[c_1rK_1(r/c_1) - c_2rK_1(r/c_2)\right]\\ &-\frac{2(y^2-3x^2)}{c_1^2-c_2^2}\left[c_1^2K_2(r/c_1) - c_2^2K_2(r/c_2)\right]\bigg\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\sigma_{yy} = -\frac{\mu b_x}{2\pi(1-\nu)}\frac{y}{r^4}\bigg\{&(y^2-x^2) - \frac{4(c_1^2+c_2^2)}{r^2}(y^2-3x^2) - \frac{2x^2}{c_1^2-c_2^2}\left[c_1rK_1(r/c_1) - c_2rK_1(r/c_2)\right]\\ &+\frac{2(y^2-3x^2)}{c_1^2-c_2^2}\left[c_1^2K_2(r/c_1) - c_2^2K_2(r/c_2)\right]\bigg\},\end{aligned} \tag{4.10}$$

$$\sigma_{xy} = \frac{\mu b_x}{2\pi(1-\nu)} \frac{x}{r^4} \left\{ (x^2 - y^2) - \frac{4(c_1^2 + c_2^2)}{r^2}(x^2 - 3y^2) - \frac{2y^2}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right.$$
$$\left. + \frac{2(x^2 - 3y^2)}{c_1^2 - c_2^2}\left[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)\right] \right\}, \tag{4.11}$$

$$\sigma_{zz} = -\frac{\mu b_x \nu}{\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right\}. \tag{4.12}$$

The trace of the stress tensor $\sigma_{kk} = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ is

$$\sigma_{kk} = -\frac{\mu b_x (1+\nu)}{\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right\}. \tag{4.13}$$

We find for the elastic strain of an edge dislocation

$$E_{xx} = -\frac{b_x}{4\pi(1-\nu)} \frac{y}{r^2} \left\{ (1-2\nu) + \frac{2x^2}{r^2} + \frac{4(c_1^2 + c_2^2)}{r^4}(y^2 - 3x^2) \right.$$
$$\left. - \frac{2(y^2 - \nu r^2)}{r^2(c_1^2 - c_2^2)}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] - \frac{2(y^2 - 3x^2)}{r^2(c_1^2 - c_2^2)}\left[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)\right] \right\}, \tag{4.14}$$

$$E_{yy} = -\frac{b_x}{4\pi(1-\nu)} \frac{y}{r^2} \left\{ (1-2\nu) - \frac{2x^2}{r^2} - \frac{4(c_1^2 + c_2^2)}{r^4}(y^2 - 3x^2) \right.$$
$$\left. - \frac{2(x^2 - \nu r^2)}{r^2(c_1^2 - c_2^2)}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] + \frac{2(y^2 - 3x^2)}{r^2(c_1^2 - c_2^2)}\left[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)\right] \right\}, \tag{4.15}$$

$$E_{xy} = \frac{b_x}{4\pi(1-\nu)} \frac{x}{r^4} \left\{ (x^2 - y^2) - \frac{4(c_1^2 + c_2^2)}{r^2}(x^2 - 3y^2) - \frac{2y^2}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right.$$
$$\left. + \frac{2(x^2 - 3y^2)}{c_1^2 - c_2^2}\left[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)\right] \right\}. \tag{4.16}$$

The dilatation is

$$E_{kk} = -\frac{b_x(1-2\nu)}{2\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \right\}. \tag{4.17}$$

Eqs. (4.9)–(4.17) are nonsingular. In fact, they are zero at $r = 0$ and have extremum values near the dislocation line. In the limit $c_2 \to 0$, we recover in Eqs. (4.9)–(4.17) the expressions calculated by Gutkin and Aifantis (1999), Gutkin (2000), Lazar (2003a), Lazar (2003b), Lazar and Maugin (in press) in gradient elasticity of Helmholtz type. In general, the extremum values of the stresses and strains depend on c_2 and c_1. The stresses Eqs. (4.9)–(4.11) are plotted for $c_2 \to c_1$ in Fig. 6. Here, we do not give the corresponding formulas in order to avoid too many long equations. But, it is not complicated to calculate the limits for the Bessel functions as we did it in the previous section. For $c_1 = c_2 = \gamma$, we have: $|\sigma_{xx}(0,y)| \simeq 0.345\mu b_x/[2\pi(1-\nu)\gamma] = 0.489\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|y| \simeq 2.10\gamma = 1.485\varepsilon$, $|\sigma_{yy}(0,y)| \simeq 0.159\mu b_x/[2\pi(1-\nu)\gamma] = 0.225\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|y| \simeq 3.102\gamma = 2.193\varepsilon$, $|\sigma_{xy}(x,0)| \simeq 0.159\mu b_x/[2\pi(1-\nu)\gamma] = 0.225\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|x| \simeq 3.102\gamma = 2.193\varepsilon$, and $|\sigma_{zz}(0,y)| \simeq 0.249\mu b_x/[\pi(1-\nu)\gamma] = 0.352\mu b_x/[\pi(1-\nu)\varepsilon]$ at $|y| \simeq 2.324\gamma = 1.643\varepsilon$. It can be seen that these stresses are smoother than the stresses obtained in first gradient elasticity of Helmholtz type. In fact, in first gradient elasticity of Helmholtz type the extremum values are: $|\sigma_{xx}(0,y)| \simeq 0.547\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|y| \simeq 0.996\varepsilon$, $|\sigma_{yy}(0,y)| \simeq 0.260\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|y| \simeq 1.494\varepsilon$, $|\sigma_{xy}(x,0)| \simeq 0.260\mu b_x/[2\pi(1-\nu)\varepsilon]$ at $|x| \simeq 1.494\varepsilon$, and $|\sigma_{zz}(0,y)| \simeq 0.399\mu b_x/[\pi(1-\nu)\varepsilon]$ at $|y| \simeq 1.114\varepsilon$. In addition, it is interesting to note that in the core region $E_{yy}(0,y)$ is significantly smaller than $E_{xx}(0,y)$ similar as in first gradient elasticity (see, e.g., Gutkin and Aifantis, 1997). They are plotted in Fig. 7.

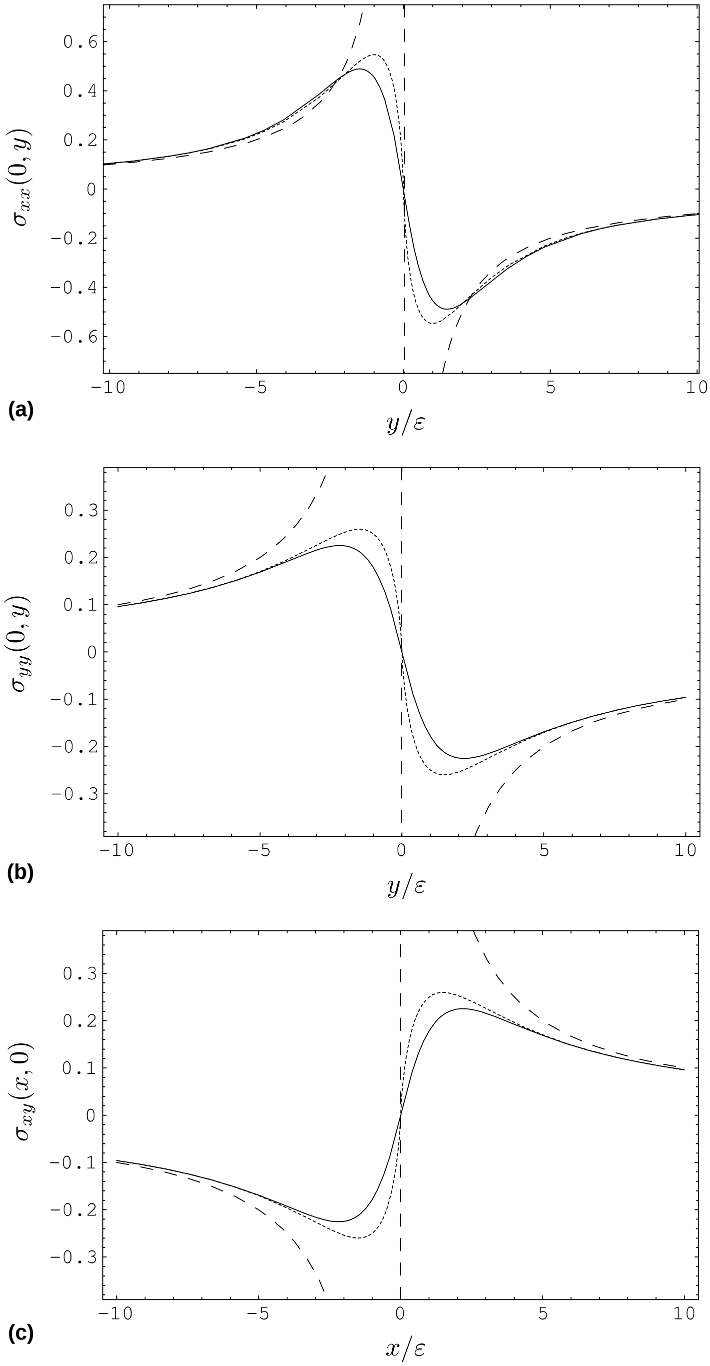

Fig. 6. Stress of an edge dislocation: (a) $\sigma_{xx}(0,y)$ and (b) $\sigma_{yy}(0,y)$ (c) $\sigma_{xy}(x,0)$ are given in units of $\mu b_x/[2\pi(1-\nu)\varepsilon]$. The full curves, small dashed curves and dashed curves, respectively, represent the stress fields in gradient elasticity of bi-Helmholtz type, gradient elasticity of Helmholtz type and classical elasticity.

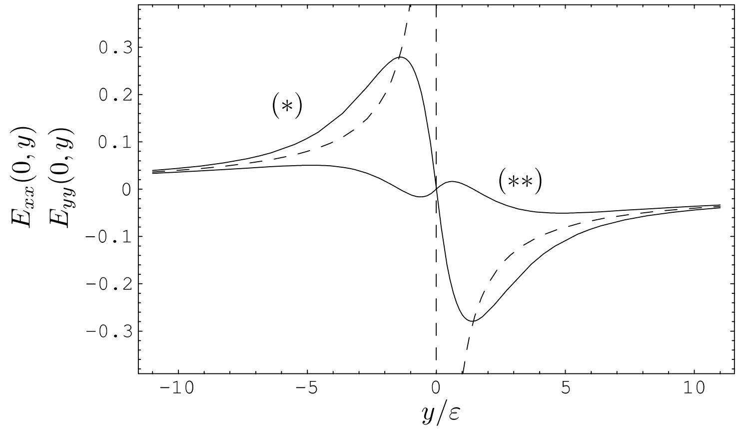

Fig. 7. Strain components $E_{xx}(0,y)$—(*) and $E_{yy}(x,0)$—(**) are given in units of $\mu b_x/[4\pi(1-\nu)\varepsilon]$ and with $\nu = 0.3$ and $c_1 = c_2$. The dashed curve represents the classical strain.

The elastic distortion is given by

$$\beta_{xx} = E_{xx}, \tag{4.18}$$
$$\beta_{xy} = E_{xy} - \omega_z, \tag{4.19}$$
$$\beta_{yx} = E_{xy} + \omega_z, \tag{4.20}$$
$$\beta_{yy} = E_{yy}, \tag{4.21}$$

where the elastic rotation, ω_z, is determined from the conditions:

$$\begin{aligned} \alpha_{xz} &= \partial_y E_{xx} + \partial_x(E_{xy} - \omega_z) = -\frac{1}{2\mu}(2\mu\partial_x\omega_z + (1-\nu)\partial_y\Delta f), \\ \alpha_{yz} &= \partial_x E_{yy} - \partial_y(E_{xy} + \omega_z) = -\frac{1}{2\mu}(2\mu\partial_y\omega_z - (1-\nu)\partial_x\Delta f) \equiv 0. \end{aligned} \tag{4.22}$$

These conditions mean that the edge dislocation has a Burgers vector in x-direction and not in y-direction. Eventually, we find for the elastic rotation

$$\omega_z = -\frac{b_x}{2\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right\}. \tag{4.23}$$

The dislocation density tensor of an edge dislocation has the following form:

$$\alpha_{xz} = \frac{b_x}{2\pi}\frac{1}{c_1^2 - c_2^2}[K_0(r/c_1) - K_0(r/c_2)]. \tag{4.24}$$

The effective Burgers vector of an edge dislocation is given by

$$b_x(r) = \oint_C (\beta_{xx}\,\mathrm{d}x + \beta_{xy}\,\mathrm{d}y) = b_x\left\{1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right\}. \tag{4.25}$$

Thus, the dislocation density (4.24) and the effective Burgers vector (4.25) of an edge dislocation have the same form as the corresponding quantities (3.23) and (3.20) of a screw dislocation. Only the tensor components are changed. Therefore, they are smooth and nonsingular even the dislocation density of a single edge dislocation. In the limits $c_2 \to 0$ and $c_1 \to 1/\kappa$, we recover in Eqs. (4.23)–(4.25) the formulas given by Lazar (2003b).

4.3. Higher order stresses

In this section, we calculate the double and triple stresses produced by an edge dislocation.

4.3.1. Double stresses

The double stresses of an edge dislocation are given in terms of the stress function f as derivatives of the third-order according to:

$$\begin{aligned}
&\tau_{(yy)x} = \varepsilon^2 \partial^3_{xxx} f, \quad \tau_{(yy)y} = -\tau_{(xy)x} = \varepsilon^2 \partial^3_{xxy} f,\\
&\tau_{(xx)y} = \varepsilon^2 \partial^3_{yyy} f, \quad \tau_{(xx)x} = -\tau_{(xy)y} = \varepsilon^2 \partial^3_{yyx} f,\\
&\tau_{(zz)x} = \nu(\tau_{(xx)x} + \tau_{(yy)x}),\\
&\tau_{(zz)y} = \nu(\tau_{(xx)y} + \tau_{(yy)y}),
\end{aligned} \tag{4.26}$$

and we obtain

$$\begin{aligned}
\tau_{(yy)x} = -\frac{\mu b_x \varepsilon^2}{2\pi(1-\nu)} \frac{2xy}{r^6} \bigg\{ &(x^2 - 3y^2) + 24 \frac{c_1^2 + c_2^2}{r^2}(y^2 - x^2) - \frac{3(y^4 - x^4)}{(c_1^2 - c_2^2) r}[c_1 K_1(r/c_1) - c_2 K_1(r/c_2)]\\
&- \frac{12(y^2 - x^2)}{c_1^2 - c_2^2}[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)] + \frac{x^2 r^2}{c_1^2 - c_2^2}[K_2(r/c_1) - K_2(r/c_2)] \bigg\}
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
\tau_{(xx)x} = -\frac{\mu b_x \varepsilon^2}{2\pi(1-\nu)} \frac{2xy}{r^6} \bigg\{ &(y^2 - 3x^2) + 24 \frac{c_1^2 + c_2^2}{r^2}(x^2 - y^2) - \frac{3(x^4 - y^4)}{(c_1^2 - c_2^2) r}[c_1 K_1(r/c_1) - c_2 K_1(r/c_2)]\\
&- \frac{12(x^2 - y^2)}{c_1^2 - c_2^2}[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)] + \frac{y^2 r^2}{c_1^2 - c_2^2}[K_2(r/c_1) - K_2(r/c_2)] \bigg\}
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\tau_{(xx)y} = -\frac{\mu b_x \varepsilon^2}{2\pi(1-\nu)} \frac{1}{r^6} \bigg\{ &(3x^4 - 6x^2y^2 - y^4) - 12 \frac{c_1^2 + c_2^2}{r^2}(x^4 - 6x^2y^2 + y^4)\\
&+ \frac{2y^4 r^2}{c_1^2 - c_2^2}[K_2(r/c_1) - K_2(r/c_2)] + \frac{6(x^4 - 6x^2y^2 + y^4)}{c_1^2 - c_2^2}[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)]\\
&- \frac{12x^2y^2}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)] \bigg\}
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\tau_{(yy)y} = \frac{\mu b_x \varepsilon^2}{2\pi(1-\nu)} \frac{1}{r^6} \bigg\{ &(x^4 - 6x^2y^2 + y^4) - 12 \frac{c_1^2 + c_2^2}{r^2}(x^4 - 6x^2y^2 + y^4)\\
&- \frac{2x^2y^2 r^2}{c_1^2 - c_2^2}[K_2(r/c_1) - K_2(r/c_2)] + \frac{6(x^4 - 6x^2y^2 + y^4)}{c_1^2 - c_2^2}[c_1^2 K_2(r/c_1) - c_2^2 K_2(r/c_2)]\\
&+ \frac{2(x^6 - 3x^4y^2 - 3x^2y^4 + y^6)}{(c_1^2 - c_2^2) r}[c_1 K_1(r/c_1) - c_2 K_1(r/c_2)] \bigg\}
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\tau_{(zz)x} = \frac{\mu b_x \nu \varepsilon^2}{\pi(1-\nu)} \frac{xy}{r^4} \bigg\{ &2\left(1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right)\\
&- r^2 \frac{1}{c_1^2 - c_2^2}[K_0(r/c_1) - K_0(r/c_2)] \bigg\},
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
\tau_{(zz)y} = -\frac{\mu b_x \nu \varepsilon^2}{\pi(1-\nu)} \bigg\{ &\frac{x^2 - y^2}{r^4}\left(1 - \frac{1}{c_1^2 - c_2^2}[c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)]\right)\\
&+ \frac{y^2}{r^2} \frac{1}{c_1^2 - c_2^2}[K_0(r/c_1) - K_0(r/c_2)] \bigg\}.
\end{aligned} \tag{4.32}$$

It is important to note that the double stresses of an edge dislocation are smooth and nonsingular in second strain gradient elasticity unlike the double stresses calculated within first strain gradient elasticity (Lazar and Maugin, in press) and first gradient micropolar elasticity (Lazar and Maugin, 2004b) which have singularities at the dislocation line. In fact, the double stresses have extremum values at the defect line or are zero at the dislocation line. The form of the double stresses of an edge dislocation is more complicated than that one of a screw dislocation. Nevertheless, (4.32) and (4.31) have a similar form as (3.31) and (3.30). In the limit $c_2 \to 0$, Eqs. (4.27)–(4.32) become singular. In addition, the elastic bend-twist of an edge dislocation is calculated as

$$\kappa_{zx} = \frac{b_x}{2\pi}\left\{\frac{x^2-y^2}{r^4}\left(1-\frac{1}{c_1^2-c_2^2}[c_1 r K_1(r/c_1)-c_2 r K_1(r/c_2)]\right)-\frac{x^2}{r^2}\frac{1}{c_1^2-c_2^2}[K_0(r/c_1)-K_0(r/c_2)]\right\}, \quad (4.33)$$

$$\kappa_{zy} = \frac{b_x}{2\pi}\frac{xy}{r^4}\left\{2\left(1-\frac{1}{c_1^2-c_2^2}[c_1 r K_1(r/c_1)-c_2 r K_1(r/c_2)]\right)-r^2\frac{1}{c_1^2-c_2^2}[K_0(r/c_1)-K_0(r/c_2)]\right\}, \quad (4.34)$$

which is nonsingular. In the limits $c_2 \to 0$ and $c_1 \to 1/\kappa$, we recover the results given by Lazar (2003b). All components of the double stress tensor and of the elastic bend-twist tensor of an edge dislocation are nonsingular and they behave like $1/r^2$ in the far field just like an electric dipole (see Fig. 8).

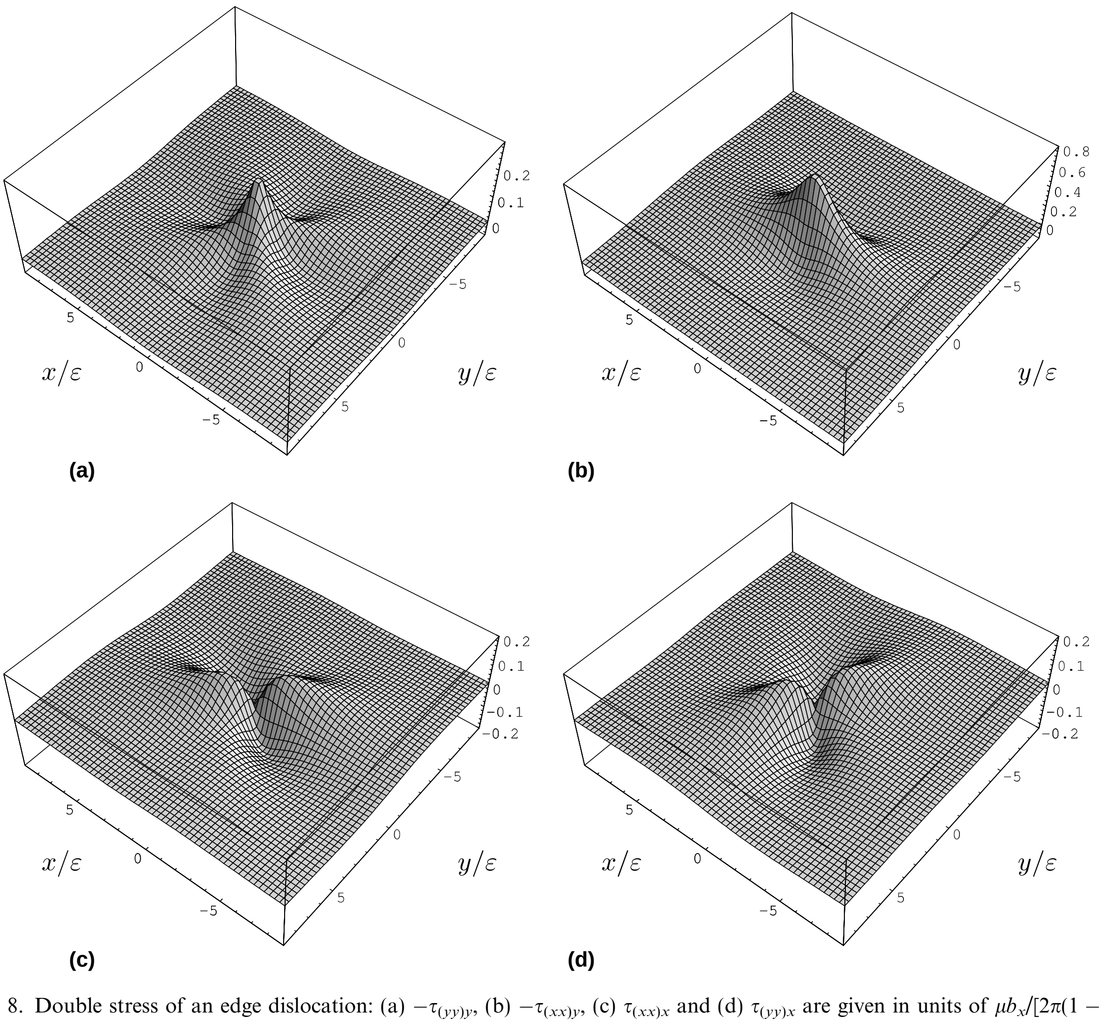

Fig. 8. Double stress of an edge dislocation: (a) $-\tau_{(yy)y}$, (b) $-\tau_{(xx)y}$, (c) $\tau_{(xx)x}$ and (d) $\tau_{(yy)x}$ are given in units of $\mu b_x/[2\pi(1-\nu)]$.

4.3.2. Triple stresses

The triple stresses of an edge dislocation are given as derivatives of fourth-order of the stress function f:

$$\begin{aligned}
&\tau_{(yy)(xx)} = \gamma^4\partial^4_{xxxx}f, \quad \tau_{(yy)(xy)} = -\tau_{(xy)(xx)} = \gamma^4\partial^4_{xxxy}f,\\
&\tau_{(xx)(yy)} = \gamma^4\partial^4_{yyyy}f, \quad \tau_{(xx)(yx)} = -\tau_{(xy)(yy)} = \gamma^4\partial^4_{yyyx}f,\\
&\tau_{(yy)(yy)} = \tau_{(xx)(xx)} = -\tau_{(xy)(xy)} = \gamma^4\partial^4_{xxyy}f,\\
&\tau_{(zz)(xx)} = \nu\big(\tau_{(xx)(xx)} + \tau_{(yy)(xx)}\big),\\
&\tau_{(zz)(yy)} = \nu\big(\tau_{(xx)(yy)} + \tau_{(yy)(yy)}\big),\\
&\tau_{(zz)(xy)} = \nu\big(\tau_{(xx)(xy)} + \tau_{(yy)(xy)}\big).
\end{aligned} \tag{4.35}$$

The result of the calculation reads

$$\begin{aligned}
\tau_{(xx)(xy)} = \frac{\mu b_x\gamma^4}{2\pi(1-\nu)}\frac{2x}{r^8}\Bigg\{&3(x^4-6x^2y^2+y^4) - 24\frac{c_1^2+c_2^2}{r^2}(x^4-10x^2y^2+5y^4)\\
&+\frac{3}{c_1^2-c_2^2}(x^6-9x^4y^2-5x^2y^4+5y^6)\left[\frac{c_1}{r}K_1(r/c_1)-\frac{c_2}{r}K_1(r/c_2)\right]\\
&+\frac{y^4r^2}{c_1^2-c_2^2}\left[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\right]+\frac{6y^2(y^4-x^4)}{c_1^2-c_2^2}[K_2(r/c_1)-K_2(r/c_2)]\\
&+\frac{12}{c_1^2-c_2^2}(x^4-10x^2y^2+5y^4)\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\Bigg\},
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\tau_{(yy)(xy)} = -\frac{\mu b_x\gamma^4}{2\pi(1-\nu)}\frac{2x}{r^8}\Bigg\{&(x^4-14x^2y^2+9y^4) - 24\frac{c_1^2+c_2^2}{r^2}(x^4-10x^2y^2+5y^4)\\
&+\frac{3}{c_1^2-c_2^2}(x^6-9x^4y^2-5x^2y^4+5y^6)\left[\frac{c_1}{r}K_1(r/c_1)-\frac{c_2}{r}K_1(r/c_2)\right]\\
&-\frac{x^2y^2r^2}{c_1^2-c_2^2}\left[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\right]\\
&+\frac{1}{c_1^2-c_2^2}(x^6-7x^4y^2-5x^2y^4+3y^6)[K_2(r/c_1)-K_2(r/c_2)]\\
&+\frac{12}{c_1^2-c_2^2}(x^4-10x^2y^2+5y^4)[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)]\Bigg\},
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\tau_{(yy)(xx)} = \frac{\mu b_x\gamma^4}{2\pi(1-\nu)}\frac{2y}{r^8}\Bigg\{&3(x^4-6x^2y^2+y^4) - 24\frac{c_1^2+c_2^2}{r^2}(5x^4-10x^2y^2+y^4)\\
&+\frac{3}{c_1^2-c_2^2}(5x^6-5x^4y^2-9x^2y^4+y^6)\left[\frac{c_1}{r}K_1(r/c_1)-\frac{c_2}{r}K_1(r/c_2)\right]\\
&+\frac{x^4r^2}{c_1^2-c_2^2}\left[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\right]+\frac{6x^2(x^4-y^4)}{c_1^2-c_2^2}[K_2(r/c_1)-K_2(r/c_2)]\\
&+\frac{12}{c_1^2-c_2^2}(5x^4-10x^2y^2+y^4)\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\Bigg\},
\end{aligned} \tag{4.38}$$

$$\tau_{(yy)(yy)} = -\frac{\mu b_x \gamma^4}{2\pi(1-\nu)}\frac{2y}{r^8}\bigg\{(9x^4-14x^2y^2+y^4) - 24\frac{c_1^2+c_2^2}{r^2}(5x^4-10x^2y^2+y^4)$$
$$+\frac{3}{c_1^2-c_2^2}(5x^6-5x^4y^2-9x^2y^4+y^6)\Big[\frac{c_1}{r}K_1(r/c_1)-\frac{c_2}{r}K_1(r/c_2)\Big]$$
$$-\frac{x^2y^2r^2}{c_1^2-c_2^2}\Big[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\Big]$$
$$+\frac{1}{c_1^2-c_2^2}(3x^6-5x^4y^2-7x^2y^4+y^6)[K_2(r/c_1)-K_2(r/c_2)]$$
$$+\frac{12}{c_1^2-c_2^2}(5x^4-10x^2y^2+y^4)\big[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\big]\bigg\}, \tag{4.39}$$

$$\tau_{(xx)(yy)} = \frac{\mu b_x \gamma^4}{2\pi(1-\nu)}\frac{2y}{r^8}\bigg\{(15x^4-10x^2y^2-y^4) - 24\frac{c_1^2+c_2^2}{r^2}(5x^4-10x^2y^2+y^4)$$
$$+\frac{3}{c_1^2-c_2^2}(5x^6-5x^4y^2-9x^2y^4+y^6)\Big[\frac{c_1}{r}K_1(r/c_1)-\frac{c_2}{r}K_1(r/c_2)\Big]$$
$$+\frac{y^4r^2}{c_1^2-c_2^2}\Big[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\Big]-\frac{2y^2}{c_1^2-c_2^2}(5x^4+4x^2y^2-y^4)[K_2(r/c_1)-K_2(r/c_2)]$$
$$+\frac{12}{c_1^2-c_2^2}(5x^4-10x^2y^2+y^4)[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)]\bigg\} \tag{4.40}$$

and

$$\tau_{(zz)(yy)} = -\frac{\mu b_x \nu \gamma^4}{\pi(1-\nu)}\frac{y}{r^6}\bigg\{(y^2-3x^2)\Big(2-r^2\frac{1}{c_1^2-c_2^2}[K_2(r/c_1)-K_2(r/c_2)]\Big)$$
$$-y^2r^2\frac{1}{c_1^2-c_2^2}\Big[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\Big]\bigg\}, \tag{4.41}$$

$$\tau_{(zz)(xy)} = \frac{\mu b_x \nu \gamma^4}{\pi(1-\nu)}\frac{x}{r^6}\bigg\{(x^2-3y^2)\Big(2-r^2\frac{1}{c_1^2-c_2^2}[K_2(r/c_1)-K_2(r/c_2)]\Big)$$
$$+y^2r^2\frac{1}{c_1^2-c_2^2}\Big[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\Big]\bigg\}, \tag{4.42}$$

$$\tau_{(zz)(xx)} = \frac{\mu b_x \nu \gamma^4}{\pi(1-\nu)}\frac{y}{r^6}\bigg\{(y^2-3x^2)\Big(2-r^2\frac{1}{c_1^2-c_2^2}[K_2(r/c_1)-K_2(r/c_2)]\Big)$$
$$+x^2r^2\frac{1}{c_1^2-c_2^2}\Big[\frac{r}{c_1}K_1(r/c_1)-\frac{r}{c_2}K_1(r/c_2)\Big]\bigg\}. \tag{4.43}$$

We plotted the components (4.36)–(4.40) in Fig. 9. Even the triple stresses are nonsingular. The components (4.41)–(4.43) are similar in the form as the components (3.40)–(3.42) of the triple stress of a screw dislocation. The triple stresses behave like $1/r^3$ in the far field just like a quadrupole.

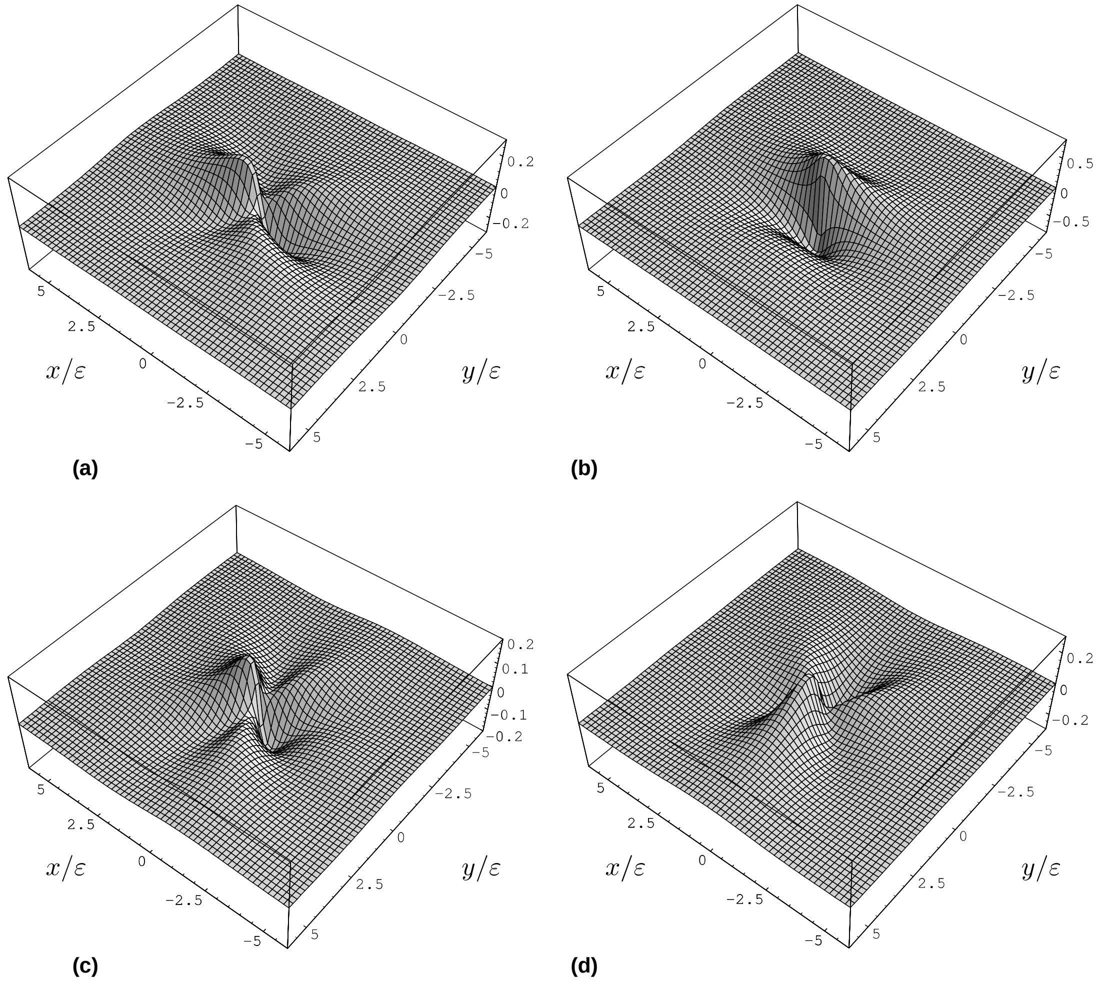

Fig. 9. Triple stress of an edge dislocation: (a) $\tau_{(xx)(xy)}$, (b) $-\tau_{(xx)(yy)}$, (c) $\tau_{(yy)(xy)}$ and (d) $\tau_{(yy)(yy)}$ are given in units of $\mu b_x \varepsilon/[8\pi(1-\nu)]$.

5. Conclusions

In this paper, we proposed a theory of second strain gradient elasticity. This theory is a generalization of first strain gradient elasticity. Such a theory contains higher order stresses like double and triple stresses. We discussed the general case of second strain gradient elasticity in addition to a simplified one which is an exceptional version. Such an exceptional case of second strain gradient elasticity is developed and used in greater detail. This version has two gradient coefficients, only. In such a version the double stress and the triple stress are given as the first and second gradients of the force stress multiplied by gradient coefficients.

The exceptional version can be connected with Eringen's nonlocal elasticity of bi-Helmholtz type. Therefore, the solutions for the force stresses are also solutions in nonlocal elasticity. As a consequence, the stresses of screw and edge dislocations calculated in this paper in the framework of second strain gradient elasticity of bi-Helmholtz type have the same form as the corresponding stress components found by Lazar et al. (in press) in the theory of nonlocal elasticity of bi-Helmholtz type. Furthermore, we discussed the new two-dimensional nonlocal kernel which is the Green function of the bi-Helmholtz equation. This kernel is nonsingular in contrast to the two-dimensional kernel of the Helmholtz equation.

Using the special version of second gradient elasticity, new exact analytical solutions for the stress, strain, distortion, dislocation density and bend-twist tensors of a straight screw dislocation and a straight edge dislocation have been found. These fields depend on the two gradient coefficients. The solutions have no singularities unlike the corresponding solutions in classical elasticity. The elimination of the singularities of the dislocation density and elastic bend-twist tensors is a new feature of second strain gradient elasticity which is not possible by means of the first strain gradient elasticity. We have used the stress function method and found the stress functions for screw and edge dislocations. The strain and force stress are zero at $r = 0$ and have their extreme values near the dislocation line like in first strain gradient elasticity. The stress and strain tensors satisfy inhomogeneous bi-Helmholtz equations. The inhomogeneous parts are given by the classical expressions. We have shown that the new solutions give in the limit from second strain gradient elasticity of bi-Helmholtz type to first strain gradient elasticity of Helmholtz type the correct expressions.

In addition, we have investigated the double and triple stresses caused by a screw dislocation and an edge dislocation. Both quantities are nonsingular. Thus, singularities of the double stresses which appear in first gradient theory are regularized and even the triple stresses do not have a singularity.

An important result in the framework of second-order gradient theory is that in the cases of screw and edge dislocations all higher order stresses are nonsingular and it was possible to remove all singularities which are still present in the first-order gradient theory. Of course, this is an unexpected and surprising result. Therefore, the second strain gradient theory is self-consistent and gives good physical results. Fortunately, because all physical state quantities are smooth and nonsingular, it is not necessary to use a third strain gradient theory which will be more complicated than the second strain gradient theory. The isotropic second strain gradient theory and all the results, which we obtained, may be used for applications in crystals which are nearly isotropic, e.g., aluminum.

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Appendix A. Green's function of the bi-Helmholtz equation

In this appendix we want to calculate the two-dimensional Green function of the bi-Helmholtz equation. The equation to be solved reads

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta)G(r) = \delta(x)\delta(y). \tag{A.1}$$

The Helmholtz-operators $(1 - c_1^2 \Delta)$ and $(1 - c_2^2 \Delta)$ are commutative. We may set

$$(1 - c_2^2 \Delta)G(r) = g \tag{A.2}$$

and

$$(1 - c_1^2 \Delta)g = \delta(x)\delta(y). \tag{A.3}$$

For the infinite space, g is given by

$$g = \frac{1}{2\pi c_1^2} K_0(r/c_1), \tag{A.4}$$

which is the two-dimensional Green function of the Helmholtz equation. Now we must solve the following inhomogeneous Helmholtz equation:

$$(1 - c_2^2 \Delta) G = \frac{1}{2\pi c_1^2} K_0(r/c_1). \tag{A.5}$$

To solve (A.5), we make the following ansatz:

$$G = C_1 K_0(r/c_1) + H \tag{A.6}$$

and obtain an equation for H

$$(1 - c_2^2 \Delta) H = \left(\frac{1}{2\pi c_1^2} - \frac{c_1^2 - c_2^2}{c_1^2} C_1 \right) K_0(r/c_1) - 2\pi c_2^2 C_1 \delta(x)\delta(y). \tag{A.7}$$

Now we set

$$H = C_2 K_0(r/c_2) \tag{A.8}$$

and find

$$C_1 = \frac{1}{2\pi} \frac{1}{c_1^2 - c_2^2}, \quad C_2 = -C_1. \tag{A.9}$$

Finally, the solution of (A.1) is given by

$$G(r) = \frac{1}{2\pi} \frac{1}{c_1^2 - c_1^2} [K_0(r/c_1) - K_0(r/c_2)]. \tag{A.10}$$

Appendix B. Stress function of the bi-Helmholtz equation for a screw dislocation

We want to solve the following inhomogeneous bi-Helmholtz equation:

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta) F = A \ln r, \quad A = \frac{\mu b_z}{2\pi}. \tag{B.1}$$

To solve (B.1), we set

$$(1 - c_2^2 \Delta) F = g \tag{B.2}$$

and

$$(1 - c_1^2 \Delta) g = A \ln r. \tag{B.3}$$

The nonsingular solution of (B.3) is given by

$$g = A\{\ln r + K_0(r/c_1)\}. \tag{B.4}$$

Substituting (B.4) into (B.2), we obtain

$$(1 - c_2^2 \Delta) F = A\{\ln r + K_0(r/c_1)\}. \tag{B.5}$$

In order to solve (B.5), we use the ansatz

$$F = C_1 \ln r + C_2 K_0(r/c_1) + F_{(1)} \tag{B.6}$$

and obtain the following equation for $F_{(1)}$:

$$(1 - c_2^2 \Delta) F_{(1)} = (A - C_1) \ln r + \left(A - \frac{c_1^2 - c_2^2}{c_1^2} C_2 \right) K_0(r/c_1) + 2\pi c_2^2 (C_1 - C_2) \delta(x)\delta(y). \tag{B.7}$$

Now we set

$$F_{(1)} = C_3 K_0(r/c_2) \quad \text{(B.8)}$$

and get

$$0 = (A - C_1)\ln r + \left(A - \frac{c_1^2 - c_2^2}{c_1^2} C_2\right) K_0(r/c_1) + 2\pi c_2^2 (C_1 - C_2 - C_3)\delta(x)\delta(y). \quad \text{(B.9)}$$

Thus, we obtain for the coefficients

$$C_1 = A, \quad C_2 = A\frac{c_1^2}{c_1^2 - c_2^2}, \quad C_3 = -A\frac{c_2^2}{c_1^2 - c_2^2}. \quad \text{(B.10)}$$

Finally, the solution of (B.1) reads

$$F = A\left\{\ln r + \frac{1}{c_1^2 - c_2^2}\left[c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)\right]\right\}. \quad \text{(B.11)}$$

It is interesting to note that (B.11) is the fundamental solution of the following PDE (bi-Helmholtz Laplace equation):

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta)\Delta F = 2\pi A\delta(x)\delta(y), \quad \text{(B.12)}$$

and (B.4) is the fundamental solution of the PDE (Helmholtz Laplace equation)

$$(1 - c_1^2 \Delta)\Delta g = 2\pi A\delta(x)\delta(y), \quad \text{(B.13)}$$

since $\Delta \ln r = 2\pi\delta(x)\delta(y)$.

Appendix C. Stress function of the bi-Helmholtz equation for an edge dislocation

Our special interest is the solution of the following inhomogeneous bi-Helmholtz equation:

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta)f = A\partial_y(r^2 \ln r), \quad A = -\frac{\mu b_x}{4\pi(1 - \nu)}. \quad \text{(C.1)}$$

In order to solve (C.1), we use the relations

$$(1 - c_2^2 \Delta)f = g \quad \text{(C.2)}$$

and

$$(1 - c_1^2 \Delta)g = A\partial_y(r^2 \ln r). \quad \text{(C.3)}$$

The nonsingular solution of (C.3) reads (see, e.g., Lazar, 2003a)

$$g = A\partial_y\{r^2 \ln r + 4c_1^2(\ln r + K_0(r/c_1))\}. \quad \text{(C.4)}$$

If we substitute (C.4) into (C.2), we have

$$(1 - c_2^2 \Delta)f = A\partial_y\{r^2 \ln r + 4c_1^2(\ln r + K_0(r/c_1))\}. \quad \text{(C.5)}$$

We use the following ansatz:

$$f = \partial_y\{r^2 \ln r + C_1 \ln r + C_2 K_0(r/c_1) + C_3 K_0(r/c_2)\} \quad \text{(C.6)}$$

and obtain

$$0 = (4c_1^2 + 4c_2^2 - C_1) + \left(4c_1^4 - \frac{c_1^2 - c_2^2}{c_1^2} C_2\right) K_0(r/c_1) + 2\pi c_2^2 (C_1 + C_2 - C_3)\delta(x)\delta(y), \tag{C.7}$$

which can be satisfied if every coefficient is zero. In this way, we find

$$C_1 = 4\left(c_1^2 + c_2^2\right), \quad C_2 = 4\frac{c_1^2}{c_1^2 - c_2^2}, \quad C_3 = -4\frac{c_2^2}{c_1^2 - c_2^2}. \tag{C.8}$$

Finally, the solution of (C.1) reads

$$f = A\partial_y \left\{ r^2 \ln r + 4(c_1^2 + c_2^2) \ln r + \frac{4}{c_1^2 - c_2^2} \left[c_1^4 K_0(r/c_1) - c_2^4 K_0(r/c_2) \right] \right\}. \tag{C.9}$$

In addition, we note that (C.9) is the fundamental solution of the following PDE (bi-Helmholtz bi-Laplace equation):

$$(1 - c_1^2 \Delta)(1 - c_2^2 \Delta)\Delta\Delta f = 8\pi A \partial_y \delta(x)\delta(y) \tag{C.10}$$

and (C.4) is the fundamental solution of the PDE (Helmholtz bi-Laplace equation)

$$(1 - c_1^2 \Delta)\Delta\Delta g = 8\pi A \partial_y \delta(x)\delta(y), \tag{C.11}$$

since $\Delta\Delta(r^2 \ln r) = 8\pi\delta(x)\delta(y)$.

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